§7.1 Integration by Substitution (U-Substitution)

Find the derivatives using the Chain Rule:

1) $\frac{d}{dx}\left(e^{4x^3}\right) =$

2) $\frac{d}{dx}\left(\tan\left(2x^5 + 7x\right)\right) =$

See if you can determine the anti-derivative for these:

3) $\int\left(24x^3\right)\sin\left(6x^4\right) =$

2) $\int\left(\frac{15x^2 + 2}{5x^3 + 2x + 3}\right)dx$

We now will find out how to "*undo*" the Chain Rule using the formal rule called U-Substitution.

Consider the problem of finding $\int(2x+3)\cos\left(x^2 + 3x\right)dx$

Step 1) Let u = __________

Step 2) Take the derivative with respect to x : $\frac{du}{dx} =$

Step 3) Rearrange: du = ___________.

Step 4) Rewrite the integral using substitution: $\int$ ________ du .

Step 5) Integrate: $\int \cos u \, du =$ __________________

Step 6) Substitute back in for u: __________________

Step 7) Check your answer by taking the derivative of the result in Step 6.

[Practice Problems]

1. $\int\left(5x^2 + 1\right)^2 (10x)\,dx$	2. $\int(1+2x)^4 (2)\,dx$	3. $\int\left(x^2 - 1\right)^3 (2x)\,dx$

4. $\int \sqrt{9-x^2}\,(-2x)\,dx$	5. $\int 2xe^{(x^2+1)}\,dx$	6. $\int 4x^3\sqrt{x^4+5}\,dx$

<u>The Substitution Rule:</u> If $u = g(x)$ is a differentiable function whose range is an interval *I* and *f* is continuous on that interval, then:

$$\int f(g(x))g'(x)dx = \int f(u)du$$

In some cases, we may need to modify $g'(x)$. Consider $\int x\left(x^2+1\right)^2 dx$

- Let $u = x^2+1$, then $du = 2xdx$, what is missing in our integrand?

[More Practice]

7. $\int x^2\sqrt{x^3+1}dx$	8. $\int \sec 2x \tan 2x dx$	9. $\int x^3\sqrt{x^4+2}dx$
10. $\int \left(1+\frac{1}{t}\right)^2 \frac{1}{t^2}dt$	11. $\int \frac{1}{1+(2x)^2}dx$	12. $\int \frac{1}{\sqrt{1-9x^2}}dx$
13. $\int x\sin x^2 dx$	14. $\int \frac{\sin\sqrt{x}}{\sqrt{x}}dx$	15. $\int \cos x \sin^2 x dx$

§7.1 More Practice with Integration by Substitution (U-Substitution)

Find the following integrals.

1. $\int e^{-3x}dx$	2. $\int \sin x \cos^3 x dx$	3. $\int \frac{1}{2x+5}dx$
4. $\int \frac{(\ln x)^2}{x}dx$	5. $\int e^x\sqrt{e^x+5}dx$	6. $\int \frac{\cos x}{2+\sin x}dx$
7. $\int \frac{x}{4-x^2}dx$	8. $\int \frac{e^{\frac{1}{y}}}{y^2}dy$	9. $\int \sin(2\theta+1)d\theta$
10. $\int \frac{\tan^2 x}{\cos^2 x}dx$	11. $\int \frac{\ln^3 x}{x}dx$	12. $\int \sqrt{x}\ (x+2)dx$

13. $\int \tan x dx$	14. $\int \cot x dx$	15. $\int \left(3\tan^5 x\right)\sec^2 x dx$
16. $\int \left(8\cot^3 x\right)\csc^2 x dx$	17. $\int \frac{\cos(3x)}{\sin^8(3x)} dx$	18. $\int \csc\left(6x^2+3x\right)\cot\left(6x^2+3x\right)\left(4x+1\right)dx$
19. $\int \left(4^{3x^5+9x}\right)\left(5x^4+3\right)dx$	20. $\int e^{5x^2+35x}\left(2x+7\right)dx$	21. $\int \sec\left(3x^4+8x\right)\tan\left(3x^4+8x\right)\left(3x^3+2\right)dx$
22. $\int \cos\left(5^{3x}\right)\left(5^{3x}\right)dx$	23. $\int \sec^2\left(\ln x^7\right)\left(\frac{1}{x}\right)dx$	24. $\int \csc^2\left(7x^3-6x^2\right)\left(7x^2-4x\right)dx$

§7.1 Integration by Substitution, the Definite Integral

Evaluate $\int_1^2 x\sin(x^2)dx$. There are two methods:

Method 1:
Change the limits of integration the indefinite integral using udu substitution to find the general antiderivative:

$$\int_1^2 x\sin\left(x^2\right)dx$$

$$\int_1^2 x\sin\left(x^2\right)dx \quad let\ \ u = x^2 \ \ \&\ \ if\ \begin{matrix} x=2 \\ x=1 \end{matrix}\ then\ \begin{matrix} u=(2)^2=4 \\ u=(1)^2=1 \end{matrix}$$

$$du = 2xdx$$

$$\frac{1}{2}\int_1^4 \sin(u)\,du$$

$$= \frac{-1}{2}\cos(u)\Big|_1^4$$

$$= \frac{-1}{2}\left[\cos(4) - \cos(1)\right] \approx 0.597$$

Method 2:
Evaluate the indefinite integral using udu substitution to find the general antiderivative:

$$\int_1^2 x\sin\left(x^2\right)dx$$

$$\int x\sin\left(x^2\right)dx \quad let\ \ u = x^2$$

$$du = 2xdx$$

$$\frac{1}{2}\int \sin(u)\,du = \frac{-1}{2}\cos(u) + c$$

$$= \frac{-1}{2}\cos\left(x^2\right)\Big|_1^2$$

$$= \frac{-1}{2}\left[\cos\left(2^2\right) - \cos\left(1^2\right)\right]$$

$$= \frac{-1}{2}\left[\cos(4) - \cos(1)\right] \approx 0.597$$

Basically:

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du$$

AP Calculus AB—Unit 7 (Chapter 7 Integration & U-Substitution)

Evaluate each integral by either method:

1. $\int_0^1 \frac{x^2}{1+x^3}\,dx$

2. $\int_1^3 \frac{(\ln z)^2}{z}\,dz$

3. $\int_1^5 x\sqrt{x^2-1}\,dx$

4. $\int_0^1 x\left(x^2+1\right)^3\,dx$

5. $\int_0^1 (3x-1)^3\,dx$

6. $\int_0^1 x\sqrt{1-x^2}\,dx$