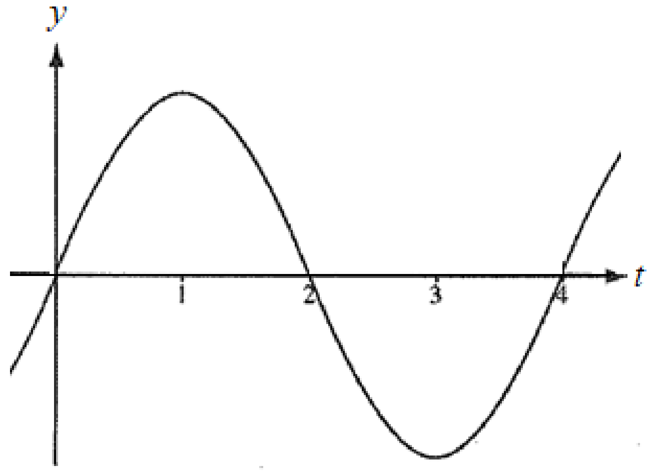
The Fundamental Theorem of Calculus - The Integral Function – class exploration
(by Benita Albert – Oak Ridge High School – Oak Ridge, Tennessee)

The graph of the function $y = f(t)$ is shown. The function is defined for $0 \le t \le 4$ and has the following properties:

- The graph of f has *odd symmetry* around the point $(2,0)$.
- On the interval $0 \le t \le 2$, the graph of f is symmetric with respect to the line $t = 1$.
- $\int_0^1 f(t)\,dt = \frac{4}{3}$



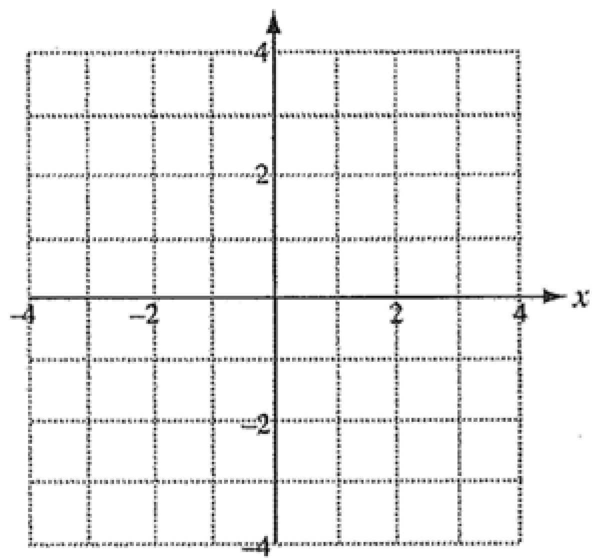
Graph of $y = f(t)$

1. Let $F(x) = \int_0^x f(t)\,dt$

a. Complete the table.

x	0	1	2	3	4
$F(x)$					

b. Plot the points from your table on the coordinate grid.



Before you sketch the curve consider what you know about the behavior of the graph of $F(x)$ based on the graph of $f(t)$. (Behavior: increasing/decreasing and concavity.)

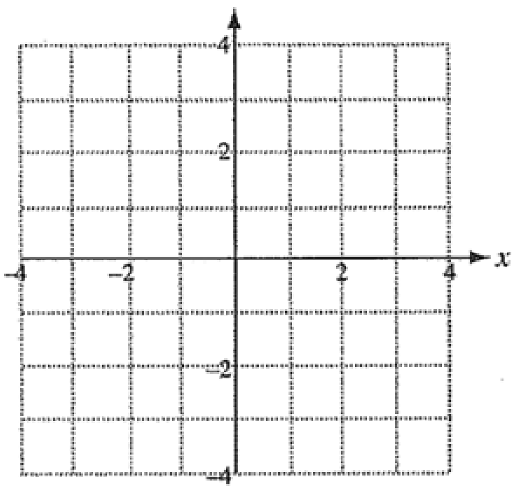
Now sketch the graph of $F(x)$.

2. Let $G(x)=\int_{2}^{x} f(t)\,dt$

a. Complete the table.

x	0	1	2	3	4
$F(x)$					

b. Plot the points from your table on the coordinate grid. Sketch the graph of $G(x)$.

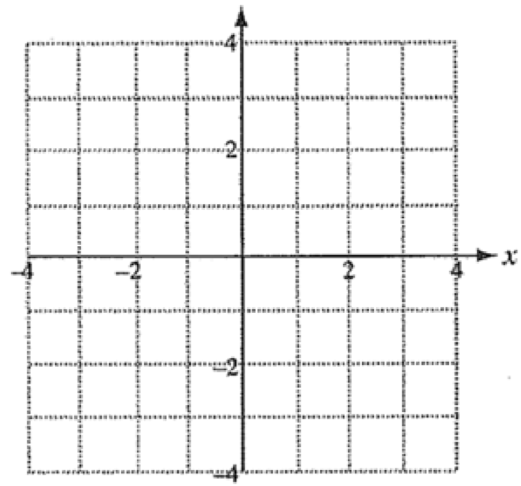


3. Let $H(x)=\int_{4}^{x} f(t)\,dt$

a. Complete the table.

x	0	1	2	3	4
$F(x)$					

b. Plot the points from your table on the coordinate grid. Sketch the graph of $H(x)$.



4. Complete the following table.

	$F(x)$	$G(x)$	$H(x)$
The maximum value of the function occurs at what x-value?			
Justify your conclusion using Calculus.			
The minimum value of the function occurs at what x-value?			
Justify your conclusion using Calculus.			
The function increases on what intervals of x?			
Justify your conclusion using Calculus.			
The function decreases on what intervals of x?			
Justify your conclusion using Calculus.			
The function is concave up on what intervals of x?			
Justify your conclusion using Calculus.			
The function is concave down on what intervals of x?			
Justify your conclusion using Calculus.			

5. What conjectures would you make about the family of functions of the form $W(x)=\int_k^x f(t)\,dt$ for $0 \le k \le 4$, where f is the graph given at the beginning of the worksheet?

The Second Fundamental Theorem of Calculus
Derivatives of Functions defined by Integrals – class exploration

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In chapter 5, we learned **The Fundamental Theorem of Calculus:**

If a function is continuous on the interval from $[a,b]$ and $f = F'$, then $\int_a^b f(t)\,dt = F(b)-F(a)$.

#1-3: For each $f(t)$, evaluate $F(x)=\int_1^x f(t)\,dt$ using the Fundamental Theorem of Calculus to find $F(x)$ in terms of x.

$1a.\quad f(t)=t^3$ $F(x)=\int_1^x t^3 dt$	$2a.\quad f(t)=4t-t^2$ $F(x)=\int_1^x 4t-t^2 dt$	$3a.\quad f(t)=\cos(t)$ $F(x)=\int_1^x \cos(t)\,dt$

Next take the derivative of each of the $F(x)$ functions you found in part 1a, 2a, and 3a above.

$1b.$	$2b.$	$3b.$

Let's reconsider #1-3 from above with a new definition: $F(x)=\int_1^{x^2} f(t)\,dt$.

Repeat the process. Use the Fundamental Theorem of Calculus to find $F(x)$ in terms of x.

1c. $f(t)=t^3$ $F(x)=\int_1^{\sin x} t^3 dt$	2c. $f(t)=4t-t^2$ $F(x)=\int_1^{\sin x} 4t-t^2 dt$	3c. $f(t)=\frac{1}{t}$ $F(x)=\int_2^{\sin x} \frac{1}{t} dt$

Take the derivative of each of the $F(x)$ functions you found in part 1c, 2c, and 3c above.

1d.	2d.	3d.

In summary:

2nd FTC 1: The Second Fundamental Theorem of Calculus part 1

If $F(x)=\int_a^{x} f(t)\,dt$ where a is a constant and f is a continuous function,

then $F'(x)=f(x)$.

2nd FTC 2: The Second Fundamental Theorem of Calculus part 2

If $F(x)=\int_a^{g(x)} f(t)\,dt$ where a is a constant and f is a continuous function, and g is a differentiable function

then $F'(x)=f(g(x))\cdot g'(x)$.

Can you use the **Second Fundamental Theorem of Calculus** to find the following derivatives <u>without</u> going through the process of anti-deriving and then deriving?

4a. $F(x)=\int_1^x 7\sqrt{t}\,dt$ $F'(x)=$	5a. $F(x)=\int_1^x \tan(t)\,dt$ $F'(x)=$	6a. $F(x)=\int_1^x \frac{1}{\sqrt[3]{t}}\,dt$ $F'(x)=$
4b. $F(x)=\int_3^{\tan x} 7\sqrt{t}\,dt$ $F'(x)=$	5b. $F(x)=\int_3^{3x^2+5x} \tan(t)\,dt$ $F'(x)=$	6b. $F(x)=\int_3^{e^x} \frac{1}{\sqrt[3]{t}}\,dt$ $F'(x)=$
4c. $F(x)=\int_2^{g(x)} 7\sqrt{t}\,dt$ $F'(x)=$	5b. $F(x)=\int_2^{h(x)} \tan(t)\,dt$ $F'(x)=$	6c. $F(x)=\int_2^{w(x)} \frac{1}{\sqrt[3]{t}}\,dt$ $F'(x)=$

7. Let $H(x)=\int_{\frac{\pi}{2}}^{x} t\cos(t)\,dt$ for $0<x<2\pi$.

a. Determine the critical number of $H(x)$.

b. Determine which critical numbers correspond to a relative maximum value of $H(x)$. Justify your answer.

c. Determine which critical numbers correspond to a relative minimum value of $H(x)$. Justify your answer.

§6.4 FTC Part II – More Practice

STATEMENT OF THE FUNDAMENTAL THEOREM OF CALCULUS PART II

If a function f is continuous on [a,b], then the function $F(x)=\int_a^x f(t)\,dt$ has a derivative at every point in (a,b) and $F'(x)=\frac{d}{dx}\left[\int_a^x f(t)\,dt\right]=f(x)$

Also note: If $F(x)=\int_a^{g(x)} f(t)\,dt$ then $F'(x)=\frac{d}{dx}\left[\int_a^{g(x)} f(t)\,dt\right]=f(g(x))\bullet g'(x)$

If $f(x)=\int_1^x t^2 dt$

then $f'(x)=$

If $f(x)=\int_1^{x^2} t^2 dt$

then $f'(x)=$

Two important things to keep in mind:

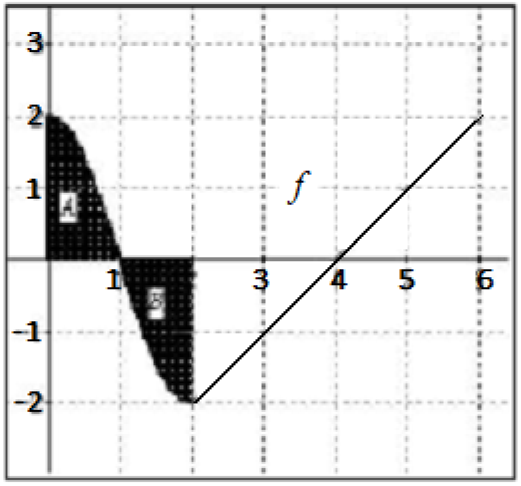
- *Always make the lower limit the constant and the upper limit the variable*
- *Multiple the answer plugged in by the derivative of the limit*

Practice: Find the derivative of each of the following functions.

1. $g(x)=\int_{-2}^{x}\sqrt{1+4t^2}\,dt$	2. $y=\int_1^x (1+t)^5 dt$	3. $F(x)=\int_0^x 3\,dt$
4. $F(x)=\int_2^x \sin t\,dt$	5. $F(x)=\int_x^1 \tan t\,dt$	6. $F(x)=\int_x^2 \cos(t^2)dt$
7. $F(x)=\int_2^{3x} 2t\,dt$	8. $F(x)=\int_\pi^{\sin x} 2t\,dt$	9. $F(x)=\int_a^x f(t)\,dt$
10. $F(x)=\int_c^{g(x)} f(t)\,dt$	11. $h(x)=\int_0^{x^2}\sqrt{1+r^3}dt$	12. $y=\int_{e^x}^0 \sin^3 t dt$
13. $h(x)=\int_1^{1/x} \ln t\,dt$	14. $g(u)=\int_{1-4u^2}^{-1}\cos t\,dt$	

Curriculum Module: Calculus: Functions Defined by Integrals

Let $F(x)=\int_{1}^{2x} f(t)\,dt$, where the graph of f on the interval $0 \le t \le 6$ is shown at the right, and the regions A and B each have an area of 1.3.



a. Compute $F(0)$ & $F(1)$.

b. Determine $F'(x)$

c. Determine the critical numbers of $F(x)$ on the interval $0 \le t \le 3$

d. Determine which critical numbers of $F(x)$ corresponds to a maximum value of $F(x)$ on the interval $0 \le t \le 3$. Justify your answer.

e. Determine which critical numbers of $F(x)$ corresponds to a minimum value of $F(x)$ on the interval $0 \le t \le 3$. Justify your answer.