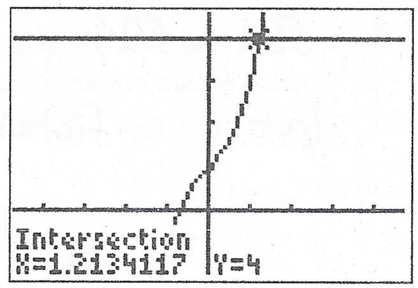
Calculus
Inverse Function Theorem

Name KEY Period 2 3 4

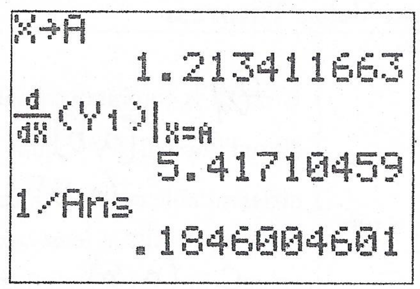
> *If f is a one-to-one differentiable function & its inverse f^{-1} is also differentiable, then for the point $f(a)=b$*
>
> $$\left(f^{-1}\right)'(b)=\frac{1}{f'(a)}$$

Example:

Let $f(x)=x^3+x+1$. Find an approximation for $\left(f^{-1}\right)'(4)$.



Solution: Because the derivative $f'(x)=3x^2+1$ is always positive, the function f is always increasing and, therefore, one-to-one. One-to-one functions have inverses even though we may not be able to solve and find the inverse function. But f has an inverse f^{-1} which is also a function.



To apply the inverse theorem, we must find the corresponding a value for the given b value of 4. Recall that the domain of a function is the range of the inverse and vice versa.

So, we want to solve for x in the equation: $x^3+x+1=4$.
Enter the function Y1= x^3+x+1 and Y2=4.

Use your calculator and the **CALC 5: Intersect** function to find the value.
The corresponding $x\approx 1.213411663...$
You can use your calculator to complete the evaluation
Store X→A. Evaluate the derivative at A. Take the reciprocal................. (See HOME screen image above).

According to the theorem above $\left(f^{-1}\right)'(4)=\dfrac{1}{f'(1.213411663...)}$. So, $\left(f^{-1}\right)'(4)\approx 0.185$.

Exercises: Record intermediate values in the table. Round all answers to the nearest thousandth.

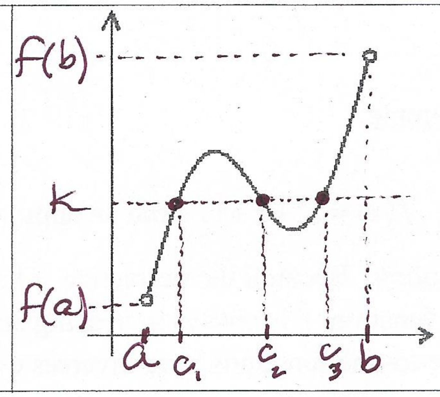
1. If $f(x)=2x+\sin x$, find $\left(f^{-1}\right)'(7)$.	$A=$ 3.80978 f(x)=7 f(x)−7=0	$f'(A)=$ 1.215056247	$\left(f^{-1}\right)'(7)=\dfrac{1}{f'(A)}=$ 0.823 0.8230071671
2. If $f(x)=x^3+9x-2$, find $\left(f^{-1}\right)'(5)$.	$A=$0.73386 f(x)=5 f(x)−5=0	$f'(A)=$ 10.61566881	$\left(f^{-1}\right)'(5)=\dfrac{1}{f'(A)}=$ 0.094 0.0942003766
3. If $f(x)=\dfrac{x^3}{x^2+1}$, find $\left(f^{-1}\right)'(2)$.	$A=$2.35930 f(x)=2 f(x)−2=0	$f'(A)=$ 1.10590645	$\left(f^{-1}\right)'(2)=\dfrac{1}{f'(A)}=$ 0.904 0.9042356162
4. If $f(x)=\dfrac{10}{x^2+1}$ for $x>0$, find $\left(f^{-1}\right)'(6)$.	$A=$0.81649 f(x)=6 f(x)−6=0	$f'(A)=$ −5.87877392	$\left(f^{-1}\right)'(6)=\dfrac{1}{f'(A)}=$ −0.170 −0.1701034952

Three Value Theorems REFRESH YOUR MEMORY of THEOREMS

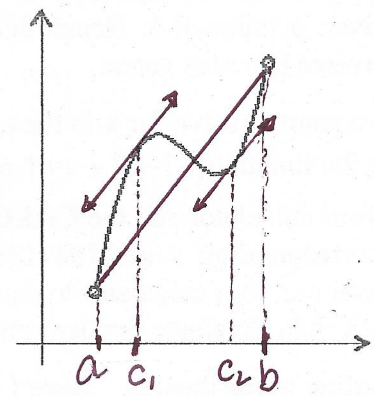
Name	Formal Statement	Restatement	Graph

FILL IN THE BLANKS ↓ DRAW PICTURE ↓

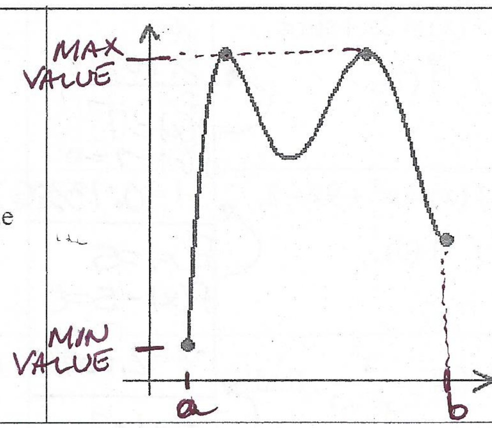
Intermediate Value Theorem

IVT	If $f(x)$ is continuous on a closed interval $[a,b]$ and k is any number between $f(a)$ and $f(b)$ then there exists at least one value in $[a,b]$ such that $f(c)=k$	On a continuous function, you will hit every y-value between two given y-values at least once.	

Mean Value Theorem

MVT	If $f(x)$ is continuous on the closed interval $[a,b]$ and differentiable on (a,b), then there must exist at least one value c in (a,b) such that $f'(c) = \frac{f(b)-f(a)}{b-a}$	If conditions are met (very important!) there is at least one point where the slope of the tangent line equals the slope of the secant line.	

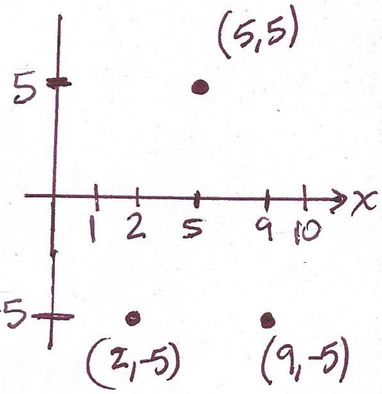
Extreme Value Theorem

EVT	A continuous function $f(x)$ on a closed interval $[a,b]$ attains both an absolute maximum $f(c) \geq f(x)$ for all x in the interval and an absolute minimum $f(c) \leq f(x)$ for all x in the interval	Every continuous function on a closed interval has a highest y-value and a lowest y-value.	

Calculus: IVT vs. MVT (vs. EVT) Name KEY

2 3 4

1. Let f be a function that is differentiable on the open interval (1, 10). If $f(2)=-5, f(5)=5$, and $f(9)=-5$, which of the following statements must be true? If the statement is true, also list which theorem guarantees it.



a. f has at least 2 zeros. $\boxed{\text{IVT}}$ since f is differentiable f(x) is continuous on (1, 10) & f(x) will hit every y-value between [-5, 5] on $x \in (1, 10)$ ∴

b. The graph has at least one horizontal tangent. $\boxed{\text{MVT}}$ Since f is diff, f is also continuous $\frac{f(9)-f(2)}{7} = f'(c)$ ∴ $f'(c)=0$ for at least one x-value on (1,10).

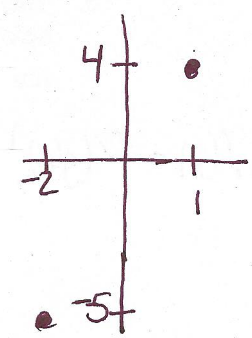
c. For some $c, 2<c<5$, $f(c)=3$ $\boxed{\text{IVT}}$ since $f(5)=5$ & $f(2)=-5$ & f(x) is continuous then $f(x)=3$ for some $c \in (2,5)$.

2. The function f is continuous for $-2 \le x \le 1$ and differentiable for $-2 < x < 1$. If $f(-2)=-5$ and $f(1)=4$, which of the following statements could be true? For each of the true statements, state which theorem supports it.

or false?

STATED

* f is continuous & differentiable.



a. There exists c, where $-2<c<1$, such that $f(c)=0$. $\boxed{\text{IVT}}$

T So $f(c)=0$ on $x \in (-2, 1)$ b/c $f(-2)=-5$ & $f(1)=4$

b. There exists c, where $-2<c<1$, such that $f'(c)=0$.

? I don't know if this is true... There is no guarantee that it is true.

c. There exists c, where $-2<c<1$, such that $f(c)=3$. $\boxed{\text{IVT}}$

T $f(-2)=-5$ & $f(1)=4$ so $f(x)=3$ for some $c \in (-2, 1)$.

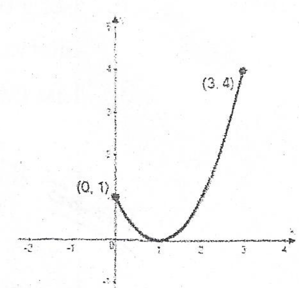
d. There exists c, where $-2<c<1$, such that $f'(c)=3$. $\boxed{\text{MVT}}$

T $\frac{f(1)-f(-2)}{1--2} = \frac{4+5}{3} = \frac{9}{3} = 3 = f'(c) = 3$ Slope of secant on (-2,1) = slope of tangent at some value $c \in (-2, 1)$.

e. There exists c, where $-2<c<1$ such that $f(c) \ge f(x)$ for all x on the closed interval $-2 \le x \le 1$.

T EXTREME VALUE THM $\boxed{\text{EVT}}$ on closed interval [-2, 1] f(x) has a max at $x=c$ where $f(c) \ge f(x)$.

3. Given the diagram at the right, that $f(x)$ is continuous, $f(0)=1$ and $f(3)=4$, is there a c on the interval (0, 3) such that $f(c)=2.5$? Justify your answer.



$f(0) < f(c) < f(3)$

$1 < 2.5 < 4$

By the $\boxed{\text{IVT}}$ since f(x) is continuous on (0, 3) $f(c)=2.5$ for some $x=c$ on the interval.

2006 AB FRQ. form B.

4. A car travels on a straight track. During the time interval $0 \le t \le 60$ seconds, the car's velocity v, measured in feet per second, and acceleration a, measured in feet per second per second, are continuous functions. The table below shows selected values of these functions.

t (sec)	0	15	25	30	35	50	60
$v(t)$ (ft/sec)	-20	-30	-20	-14	-10	0	10
$a(t)$ (ft/sec^2)	1	5	2	1	2	4	2

a. For $0 < t < 60$, must there be a time t when $v(t) = -5$? Justify your answer. $\boxed{\text{IVT}}$ applies b/c $v(t)$ is continuous.

$v(35) < v(t) < v(50)$

$-10 < v(t) < 0$

so there must be a value $v(t) = -5$ since $-10 < -5 < 0$.

b. For $0 < t < 60$, must there be a time t when $a(t) = 0$? Justify your answer.

$a(t) = v'(t)$ is continuous and differentiable

$\frac{v(25) - v(0)}{25 - 0} = \frac{-20 - -20}{25} = 0 \therefore a(t) = v'(t) = 0$ by the $\boxed{\text{MVT}}$.

5. The functions f and g are differentiable for all real numbers, and g is strictly increasing. The table below gives the values of the functions and their first derivatives at the selected values of x. The function h is given by $h(x) = f(g(x)) - 6$.

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
1	6	4	2	5
2	9	2	3	1
3	10	-4	4	2
4	-1	3	6	7

a. Explain why there must be a value r for $1 < \boxed{r} < 3$ such that $h(r) = -5$.

$h(1) = f(g(1)) - 6 = f(2) - 6 = 9 - 6 = 3$

$h(3) = f(g(3)) - 6 = f(4) - 6 = -1 - 6 = -7$

Because $h(x)$ is a composition of $f(g)$, it is differentiable & by $\boxed{\text{IVT}}$ $h(3) < -5 < h(1)$ $-7 < -5 < 3 \therefore$ there is a value r, such that $1 < r < 3$ where $h(r) = 5$.

b. Explain why there must be a value c for $1 < c < 3$ such that $h'(c) = -5$.

$h(x)$ is continuous on $[1,3]$ & differentiable on $(1,3)$ so by $\boxed{\text{MVT}}$

since $\frac{h(3) - h(1)}{3 - 1} = \frac{-7 - 3}{2} = -5 = h'(c)$ there must be a value c for which $h'(c) = 5$.

6. Let g be a continuous function on the closed interval $-1 \le x \le 3$ and differentiable on the open interval $-1 < x < 3$. If $g(-1) = -10$ and $g(3) = 6$, which of the following are guaranteed? List the theorem that guarantees those statements as well.

$\frac{g(3) - g(-1)}{3 - -1} = \frac{6 - -10}{4} = \frac{16}{4} = 4$

a. $g'(c) = 0$ for some c in the interval $-1 < x < 3$.

FALSE no guarantee that $g'(c) = 0$ on $(1,3)$

b. $g'(c) = 4$ for some c in the interval $-1 < x < 3$. $g(x)$ is continuous on $[1,3]$ & diff on $(1,3)$ & the $\boxed{\text{MVT}}$ so there must be a c value on $(1,3)$.

TRUE $\boxed{\text{MVT}}$

$\underbrace{\frac{g(3) - g(-1)}{3 + 1} = 4 = g'(c)}_{\text{is guaranteed by}}$

c. $g(c) = 4$ for some c in the interval $-1 < x < 3$.

TRUE

$g(-1) < g(c) < g(3)$ & $g(x)$ is continuous on $[1,3]$

$-10 < g(c) < 6$

$\therefore g(c) = 4$ for some value of c on $(1,3)$ by the $\boxed{\text{IVT}}$.