Part 2a: Volumes by Cross-Sections on a Base

Record the area formulas for these shapes.

To find the volume of a figure with a known cross sectional area, use the Riemann sum that becomes $\int_Area\,dx$ for a function.

1. The base of a solid is bound by $y = \sqrt{x}$, $y = 0$, and $x = 4$.

   a) Draw a picture of this base on each of the four the grids at the right.

   b) Draw three segments in this base perpendicular to the x-axis. Repeat for the next three grids at the right.

   c) What is the length of one representative segment in terms of $x$?

   d) The segments you drew in part b)

      o on the 1st grid are sides of a **square**.
      What is the area of the square in terms of $x$?
      \[ S = \sqrt{x}, \quad A = (\sqrt{x})^2 = x \]

      o on the 2nd grid are legs of a **right isosceles triangle**.
      What is the area of the right isosceles triangle in terms of $x$?
      \[ S = \sqrt{x}, \quad A = \frac{1}{2}(\sqrt{x})^2 = \frac{1}{2}x \]

      o on the 3rd grid are diameters of a **semi-circle**.
      What is the area of the semi-circle in terms of $x$?
      \[ r = \frac{1}{2}x, \quad A = \frac{1}{2}(\pi \left(\frac{1}{2}x\right)^2) = \frac{\pi}{8}x \]

      o on the 4th grid are sides of an **equilateral triangle**.
      What is the area of the equilateral triangle in terms of $x$?
      \[ S = \sqrt{x}, \quad A = \frac{\sqrt{3}}{4}(\sqrt{x})^2 = \frac{\sqrt{3}}{4}x \]

   e) Write an integral to find the volume formed by the solid with the given base area if the cross-section areas perpendicular to the x-axis are $\int A(x)\,dx$.

<table>
<thead>
<tr>
<th>Squares</th>
<th>Right Isosceles triangles</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\int_0^4 x,dx = 8$</td>
<td>$\int_0^4 \frac{1}{2}x,dx = 4$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Semi-Circles</th>
<th>Equilateral triangles</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\int_0^4 \frac{\pi}{2}x,dx = \frac{\pi}{2}$</td>
<td>$\int_0^4 \frac{3\sqrt{3}}{4}x,dx = 2\sqrt{3}$</td>
</tr>
</tbody>
</table>

f) What would change if the cross-sections were perpendicular to the y-axis?

| dy | thickness
|---|---
| $y^2 = x$ |

4. Find the volume formed by the solid whose side length is bound by the base area and whose cross-section areas perpendicular to the y-axis are i) squares, ii) right isosceles triangles, iii) semi-circles, iv) equilateral triangles.

i) $\int_0^4 (4-y^2)\,dy$ ii) $\int_0^4 \frac{1}{2}(4-y^2)\,dy$ iii) $\int_0^4 \frac{\pi}{2}(4-y^2)\,dy$ iv) $\int_0^4 \frac{\sqrt{3}}{4}(4-y^2)\,dy$
Part 2b: Volumes of Revolution (Disks)

Example 1: The region between the curve \( y = \sqrt{x}, \ 0 \leq x \leq 4 \) and the x-axis is revolved around the x-axis to generate a solid. Find its volume.

Follow these steps to visualize the volumes and process the algebraic solution:
- Draw the bounded region described.
- Reflect the region in the x-axis.
- Draw three representative circular slices or disks perpendicular to the axis of rotation.
- Identify the radius of each disk in terms of the variable \( x \).
- Identify the area of each disk in terms of the variable \( x \).
- Disks perpendicular to a horizontal axis of rotation will have a dx-thickness.
- Identify the volume of each disk in terms of the variable \( x \).
- Write an integral to sum the volume of all the disks.

Show work here:

\[
\begin{align*}
V &= \pi \int_{0}^{4} \pi x \, dx \\
&= \pi \left[ \frac{x^2}{2} \right]_{0}^{4} \\
&= \frac{\pi}{2} \left( 4^2 - 0^2 \right) \\
&= 8\pi
\end{align*}
\]

Example 2: The region between the curve \( y = \sqrt{4 - x}, \ 0 \leq x \leq 4 \) and the y-axis is revolved around the y-axis to generate a solid. Find its volume.

Follow the steps outlined above. Since the region is revolved about the y-axis the disks are now perpendicular to the y-axis and they have a dy-thickness.

Show work here:

\[
\begin{align*}
V &= \pi \int_{0}^{2} (4 - y^2)^2 \, dy \\
&= \pi \int_{0}^{2} (16 - 8y^2 + y^4) \, dy \\
&= \pi \left[ 16y - \frac{8}{3}y^3 + \frac{1}{5}y^5 \right]_{0}^{2} \\
&= \pi \left( 32 - \frac{64}{3} + \frac{32}{5} \right) \\
&= 32\pi \left( 1 - \frac{2}{3} + \frac{1}{5} \right) \\
&= \frac{224\pi}{5}
\end{align*}
\]

Volume of Revolution by Disks

Volume of a solid obtained by revolving a bounded region about an axis of rotation:

\[
V = \int_{a}^{b} \pi [R(x)]^2 \, dx \quad \text{or} \quad V = \int_{c}^{d} \pi [R(y)]^2 \, dy
\]

for \( a \leq x \leq b \) \quad \text{for} \quad c \leq y \leq d
Problems: Graph the region. Write and evaluate an integral to find the volume of the solid generated by revolving the region about the stated axis. #1 & #8 - do by hand. #2-6 - evaluate with a calculator.

1. \( y = e^{-x}, \ y = 0, \ x = 1, \ x = \ln(10) \) 
   \( x \)-axis 
   \[ r = y = e^{-x}, \quad A = \pi \left( e^{-2x} \right) \]

2. \( y = \sqrt{x}, \ y = 0, \ x = 4 \) 
   \( x \)-axis: \( x = 4 \) 
   \[ r = y = 4 - \sqrt{x}, \quad A = \pi \left( 4 - y^2 \right) \]

3. \( y = x^2, \ y = 4 \) 
   \( x \)-axis: \( y = 4 \) 
   \[ r = 4 - y = (4 - x^2), \quad A = \pi \left( 4 - y^2 \right) \]

4. 9x^2 + 25y^2 = 225 
   \[ \frac{x^2}{25} + \frac{y^2}{9} = 1 \] 
   \( A = \pi \left( \frac{2}{3} \sqrt{25-x^2} \right) \]

5. \( y = 4 - x^2, \ y = 0, \ x = -2, \ x = 0 \) 
   \( x \)-axis 
   \[ r = y = 4 - x^2 \]

6. \( y = \cos(x), \ y = 0, \ x = 0, \ x = \frac{\pi}{2} \) 
   \( x \)-axis 
   \[ A = \pi \left( \cos^2 x \right) = \pi \cos^2 x \]

\[ V = \pi \int_0^{\frac{\pi}{2}} \cos^2 x \, dx = 2.467 \]
7. What is the formula for the volume of a sphere?

\[ V = \frac{4}{3}\pi r^3 \]

FYE.

\[ \frac{dV}{dr} = 4\pi r^2 = \text{SA} \]

8. \( a \) circle with radius "r" on the y-axis

\[ r = x = \sqrt{r^2 - y^2} \]

\[ A_y = \pi r^2 = \pi \left( r^2 - y^2 \right) \]

\[ V = \pi \int_{-r}^{r} \left( r^2 - y^2 \right) dy \]

\[ V = 2\pi \left( r^2 y - \frac{1}{3} y^3 \right) \bigg|_{-r}^{r} = 2\pi \left( r^3 \right) \]

\[ = \frac{4\pi}{3} r^3 \]

---

**DAY 117 HW** Volumes of Revolution by Disks: #1-2 below and AP FRQ #3-4-5 next page.

1. (Non-Calculator) Write and evaluate an integral to find the volume of the solid formed by revolving the shaded region in the X-AXIS. Show all work.
Part 2c: Volumes of Revolution (Washers)

Sometimes slicing a solid of revolution results in disks with holes in the middle—washers. In order to find these volumes, it is necessary to subtract the volume of the hole.

$$A = \pi R^2 - \pi r^2$$
$$V = \pi \int (R^2 - r^2) \, dx$$
$$V = \pi \int (R^2 - r^2) \, dy$$

**Example 1:** The region between the curve $y = \sqrt{x}$, $0 \leq x \leq 4$ and the x-axis is revolved around the horizontal line $y = -2$ to generate a solid. Find its volume.

Follow these steps to visualize the volumes and process the algebraic solution:

- Draw the bounded region described. Draw the axis of rotation.
- Reflect the region in the axis of rotation.
- Draw three representative circular washer slices perpendicular to the axis of rotation.
- **Identify the Outer-R and Inner-r radius** of each washer in terms of the variable $x$.
- **Identify the area** of each washer in terms of the variable $x$.
- Washers perpendicular to a horizontal axis of rotation will have a $dx$-thickness.
- **Identify the volume** of each washer in terms of the variable $x$.
- **Write an integral** to sum the volume of all the washers.

Show work here:

$$R = y + 2$$
$$r = 0 - 2$$
$$dx - thickness : R = \sqrt{x} + 2$$

$$\pi \int R^2 - r^2 = \pi \left( (\sqrt{x} + 2)^2 - (2)^2 \right)$$
$$= \pi \left( \sqrt{x}^2 + 4 \sqrt{x} + 4 - 4 \right)$$
$$= \pi \left( x + 4 \sqrt{x} + 0 - 0 \right) = \pi \left( x + 4 \sqrt{x} \right)$$

$$V = \pi \int_0^4 (x + 4 \sqrt{x}) \, dx$$

$$V = \pi \left( \frac{x^2}{2} + \frac{8}{3} x^{3/2} \right) \bigg|_0^4$$

$$= \pi \left( \left( 8 + \frac{64}{3} \right) - (0) \right) = \frac{88 \pi}{3} \approx 92.153$$
Example 2: The region between the curve \( y = \sqrt{x}, \ 0 \leq x \leq 4 \) and the x-axis is revolved around the vertical line \( x = -1 \) to generate a solid. Find its volume.

Follow the steps outlined above. Since the region is revolved about the vertical line-axis the washers are now perpendicular to the y-axis and they have a \( dy \)-thickness.

Show work here:

\[
\begin{align*}
\pi \int_0^4 (4 - y^2) \, dy = 4 \\
\pi \int_0^4 16 - y^4 \, dy = 256 \pi \\
2 \pi \left( 16y - \frac{1}{5} y^5 \right) \bigg|_0^4 = 128 \pi \\
\pi \left( 16 \omega - \frac{8}{5} \right) = \frac{128 \pi}{5}
\end{align*}
\]

Volume of Revolution by Washers

Volume of a solid obtained by revolving a bounded region about an axis of rotation:

- about a horizontal axis:
  \[ V = \pi \int_a^b \left( [R(x)]^2 - [r(x)]^2 \right) \, dx \]
- about a vertical axis:
  \[ V = \pi \int_c^d \left( [R(y)]^2 - [r(y)]^2 \right) \, dy \]

for \( a \leq x \leq b \)

for \( c \leq y \leq d \)

Problems:

Draw a picture and find the volume of the solid found by revolving the region about the stated axis.

\[ \begin{align*}
1. \quad & y = 2 - x^2, \ y = 1 \quad (R = y = 2 - x^2) \\
\text{x-axis} & \quad (r = 1) \\
\text{dx-thickness} & \\
& \quad V = \pi \int_{-1}^1 \left( 2 - x^2 \right) - 1^2 \, dx \\
& \quad = \frac{2 \pi}{3} \\
& \quad = \frac{2 \pi}{3} \\
2. \quad & y = 2 - x^2, \ y = 1 \quad (R = y = 2 - x^2) \\
\text{R} & \quad y = -3 \\
\text{dx-thickness} & \quad (r = 4) \\
& \quad V = \pi \int_{-1}^1 \left( 2 - x^2 \right) - 1^2 \, dx \\
& \quad = \frac{2 \pi}{3} \\
& \quad = \frac{2 \pi}{3} \\
3. \quad & y = 2 - x^2, \ y = 1 \\
\text{R} & \quad y = 3 \\
\text{dx-thickness} & \quad (r = 3) \\
& \quad V = \pi \int_{-1}^1 \left( 2 - x^2 \right) - 1^2 \, dx \\
& \quad = \frac{2 \pi}{3} \\
& \quad = \frac{2 \pi}{3} \\
\end{align*} \]
4. \( y = \frac{1}{x}, \quad y = \sqrt{x}, \quad x = 1, \quad x = 4 \)

\[ R = \sqrt{x} \]

\[ \pi \int_1^4 (\sqrt{x})^2 - \frac{1}{x^2} \, dx \]

\[ \pi \int_1^4 x - \frac{1}{x^2} \, dx \]

\[ \pi \left( \frac{x^3}{3} + \frac{1}{x} \right) \bigg|_1^4 \]

\[ \pi \left( 8 + \frac{1}{4} \right) - \left( \frac{1}{4} + 1 \right) \]

\[ \frac{25\pi}{4} \approx 20.4156 \]

5. \( y = \frac{1}{x}, \quad y = \sqrt{x}, \quad x = 1, \quad x = 4 \)

\[ R = 2 - y = 2 - \frac{1}{x} \]

\[ \int_1^4 (2 - \frac{1}{x})^2 - (2 - \sqrt{x})^2 \, dx \]

\[ (4 - \frac{4}{x} + \frac{1}{x^2}) - (4 - 4\sqrt{x} + x) \]

\[ \pi \int_1^4 \frac{4\sqrt{x} - x - \frac{4}{x} + \frac{1}{x^2}}{x} \, dx \]

\[ \pi \left( \frac{8x^2}{3} - \frac{1}{3} x^2 - 4 \ln|x| \right) - \frac{1}{x} \bigg|_1^4 \]

\[ \frac{\pi}{12} \left[ 8 - 4 \ln 4 \ln 4 - \frac{1}{4} \right] - \left( \frac{8}{3} - \frac{1}{4} \right) \]

\[ \frac{20.0146}{20.017} \]

6. The region in the first quadrant bounded by \( y = \sqrt{6x + 4} \) & \( y = 2x \) when revolved about the \( x \)-axis.

\[ \pi \int_0^2 \left( \frac{(6x+4)^2}{(2x+2)^2} - (2x)^2 \right) \, dx \]

\[ \pi \int_0^2 6x+4 - 4x^2 \, dx \]

\[ \pi \left( 3x^2 + 4x - \frac{4}{3} x^3 \right) \bigg|_0^2 \]

\[ \pi \left( 12 + 8 - \frac{32}{3} \right) - (0) \]

\[ \frac{28\pi}{3} \approx 29.3215 \frac{\pi}{2} \approx 29.3215 \]

7. The region bounded by \( y = x^3, \quad y = 0 \) & \( x = 1 \) when revolved about

a) the \( y \)-axis

\[ \pi \int_0^1 1^2 - (3\sqrt[3]{y})^2 \, dy \]

\[ \pi \int_0^1 (1 - y^{\frac{3}{2}}) \, dy \]

\[ \pi \left( y - \frac{2}{3} y^{\frac{5}{2}} \right) \bigg|_0^1 \]

\[ \pi \left( 1 - \frac{2}{3} \right) \]

\[ \frac{2\pi}{5} \approx 1.2566 \]

b) \( x = -1 \)

\[ \pi \int_0^1 2^2 - (2\sqrt[3]{y} + 1)^2 \, dy \]

\[ \pi \int_0^1 4 - (y^{\frac{3}{2}} + 2y^{\frac{3}{2}} + 1) \, dy \]

\[ \pi \left( 3 - y^{\frac{3}{2}} - 2y^{\frac{3}{2}} \right) \bigg|_0^1 \]

\[ \pi \left( \frac{3\sqrt[3]{y} - \frac{3}{2} y^{\frac{3}{2}} - \frac{3}{2} y^{\frac{3}{2}} \right) \bigg|_0^1 \]

\[ \pi \left( 3 - \frac{3}{2} \right) \left( -\frac{3}{2} \right) \left( -\frac{3}{2} \right) \bigg|_0^1 \]

\[ \pi \left( 30 - 15 - 15 \right) \frac{\pi}{10} \]

\[ \frac{\pi}{10} = 0.31415 \]

c) \( x = 1 \)

\[ \pi \int_0^1 (1 - y^{\frac{3}{2}})^2 \, dy \]

\[ \pi \int_0^1 (1 - 2y^{\frac{3}{2}} + y^{\frac{3}{2}}) \, dy \]

\[ \pi \left( y - \frac{2}{3} y^{\frac{3}{2}} + \frac{2}{3} y^{\frac{3}{2}} \right) \bigg|_0^1 \]

\[ \pi \left( \left( 1 - 3\sqrt[3]{y} + 3\sqrt[3]{y} \right) \left( -\frac{3}{2} \right) \left( -\frac{3}{2} \right) \right) \bigg|_0^1 \]

\[ \pi \left( \frac{10 - 15 + 6}{10} \right) \]

\[ \frac{\pi}{10} = 0.31415 \]
Suppose we wish to calculate the total mass or population but the density is not constant over the region, we must divide the region into smaller pieces in such a way that the density is approximately constant on each piece.

To calculate the total mass or total population where the density varies over the region,

- divide the region into small pieces of relatively constant density
- add up the contributions of all the pieces
- always use units in the setups.

Example: Quadville is a city in the shape of a rectangle, five miles on one side and six miles on the other. A highway runs along the side that is six miles long. The population density \( x \) miles from the highway is given by \( p(x) = 20 - 4x \) thousand people per square mile. What is the approximate population of Quadville?

Solution:
1. Dimensional Analysis: What are the units involved in the scenario and what are the units of the answer?
   \[
   \text{(thousand people)} \cdot \text{(square mile)} = \text{thousands of people}
   \]
2. Draw a diagram including the "slices":
   \[
   \text{Area} = \int_0^6 \text{dx}
   \]
3. Determine the population of each representative slice:
   \[
   \text{Population} = (20-4x)(6) \text{dx}
   \]
4. Determine the total population:
   \[
   \text{Total} = \int_0^5 (20-4x)(6) \text{dx}
   \]

Ex 1: The population density in Ringsburg is a function of the distance from the city center. Suppose that at \( r \) miles from the center, the density is given by the function \( p(r) = \sqrt{36 - r^2} \) in thousand persons/mile\(^2\). Ringsburg has a radius of 5 miles. Estimate the total population of Ringsburg.

Ex 2: The air density (in kg/m\(^3\)) \( h \) meters above the earth's surface is \( P = f(h) \). Find the mass of a cylindrical column of air 12 meters in diameter and 25 kilometers high if \( f(h) = 1.28e^{-0.00012h} \). Write the Riemann sum and definite integral and then evaluate. (Hint: Be careful with units.)
Ex 3: A city is built around a circular lake that has a radius of 1 mile. The population density of the city is \( f(r) \) people per square mile, where \( r \) is the distance from the center of the lake, in miles. Which of the following expressions gives the number of people who live within 1 mile of the lake?

(A) \( 2\pi \int_0^1 r f(r) \, dr \)  
(B) \( 2\pi \int_0^1 r (1 + f(r)) \, dr \)  
(C) \( 2\pi \int_0^1 r (1 + f(r)) \, dr \)  
(D) \( 2\pi \int_1^2 r f(r) \, dr \)  
(E) \( 2\pi \int_1^2 r (1 + f(r)) \, dr \)

Ex 4: A city located beside a river has a rectangular boundary as shown in the figure above. The population density of the city at any point along a strip \( x \) miles from the river’s edge is \( f(x) \) people per square mile. Which of the following expressions gives the population of the city?

(A) \( \int_0^4 f(x) \, dx \)  
(B) \( 7 \int_0^4 f(x) \, dx \)  
(C) \( 28 \int_0^4 f(x) \, dx \)  
(D) \( \int_0^7 f(x) \, dx \)  
(E) \( 4 \int_0^7 f(x) \, dx \)

Page 447 #19 A semi-cylindrical storage shed is shown below has radius \( r \) and length \( l \).

a. What is the volume of the shed? 
\[
V = \left( \frac{1}{2} \pi r^2 \right) l
\]

b. The shed is filled with sawdust whose density (mass/unit volume) at any point is proportional to the distance of that point from the floor. The constant of proportionality is \( k \). Calculate the total mass of the sawdust.

\[
\text{density is proportional to distance from floor } d \text{, mass } = (\text{density})(\text{volume}) = (ky)
\]
\[
mass = \int_0^r (ky)(2\sqrt{r^2 - y^2}) \, dy
\]
\[
= 2kl \int_0^r (y \sqrt{r^2 - y^2}) \, dy
\]
\[
= k \int_0^r \sqrt{u} \, du
\]
\[
= k \int_0^{r^2} \sqrt{u} \, du = k \frac{2}{3} \left( \frac{u^{3/2}}{3} \right)_0^{r^2} = k \frac{2}{3} (r^3 - 0) = \frac{2klr^3}{3}
\]