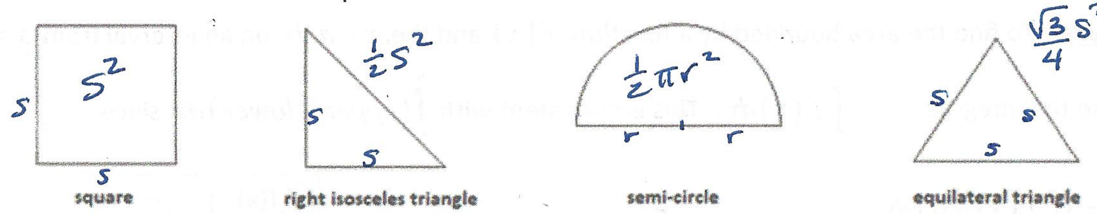
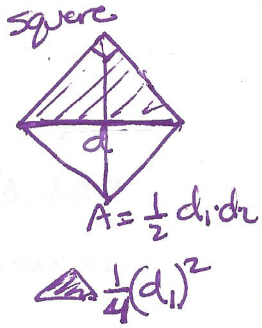
NOTES DAY 116

Part 2a: Volumes by Cross-Sections on a Base

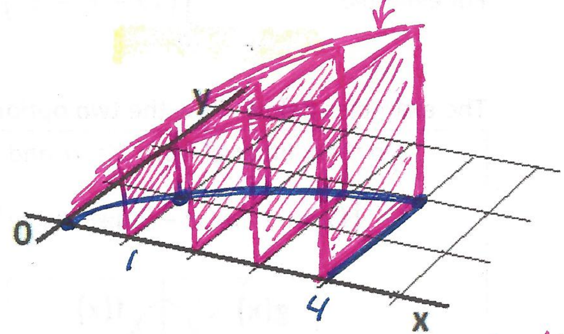
Record the area formulas for these shapes.





To find the volume of a figure with a known cross sectional area, use the Riemann sum that becomes $\int Area\, dx$ for a function.

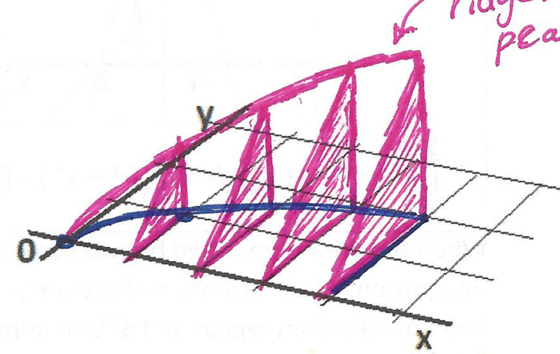
1. The base of a solid is bound by $y=\sqrt{x},\ y=0,$ and $x=4$.



a) Draw a picture of this base on each of the four the grids at the right.

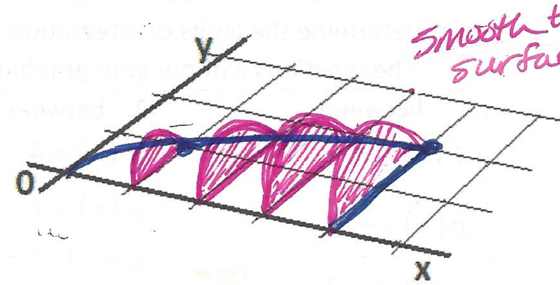
b) Draw three segments in this base perpendicular to the x-axis. Repeat for the next three grids at the right.

c) What is the length of one representative segment in terms of x?



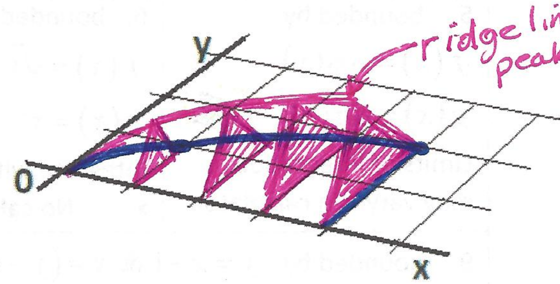
d) The segments you drew in part b)

- on the 1st grid are sides of a square.
  What is the area of the square in terms of x?
  $S=\sqrt{x}$ $\quad A=(\sqrt{x})^2=x$
- on the 2nd grid are legs of a right isosceles triangle.
  What is the area of the right isosceles triangle in terms of x?
  $S=\sqrt{x}$ $\quad A=\frac{1}{2}(\sqrt{x})^2=\frac{1}{2}x$
- on the 3rd grid are diameters of a semi-circle.
  What is the area of the semi-circle in terms of x?
  $r=\frac{1}{2}\sqrt{x}$ $\quad A=\frac{1}{2}(\pi(\frac{1}{2}\sqrt{x})^2)=\frac{\pi}{8}x$
- on the 4th grid are sides of an equilateral triangle.
  What is the area of the equilateral triangle in terms of x?
  $S=\sqrt{x}$ $\quad A=\frac{\sqrt{3}}{4}(\sqrt{x})^2=\frac{\sqrt{3}x}{4}$



e) Write an integral to find the volume formed by the solid with the given base area if the cross-section areas perpendicular to the x-axis are Volume of one cross-section $=(A)(dx)$

Squares $\int_0^4 x\,dx=8$	Right Isosceles triangles $\int_0^4 \frac{1}{2}x\,dx=4$
Semi-Circles $\int_0^4 \frac{\pi}{8}x\,dx=\pi$	Equilateral triangles $\int_0^4 \frac{\sqrt{3}}{4}x\,dx=2\sqrt{3}$



$\boxed{y^2=x}$

f) What would change if the cross-sections were perpendicular to the y-axis? dy - thickness $\quad s=4-x=4-y^2$

g) Find the volume formed by the solid whose side length is bound by the base area and whose cross-section areas perpendicular to the y-axis are i) squares, ii) right isosceles triangles, iii) semi-circles, iv) equilateral triangles.

i) $\int_0^2 (4-y^2)^2 dy$ $\quad$ 17.0666

ii) $\int_0^2 \frac{1}{2}(4-y^2)^2 dy$ $\quad$ 8.53333

iii) $\int_0^2 \frac{\pi}{2}\left(\frac{4-y^2}{2}\right)^2 dy$ $\quad$ $2.13333\pi=6.70206$

iv) $\int_0^2 \frac{\sqrt{3}}{4}(4-y^2)^2 dy$ $\quad$ $\sqrt{3}\cdot 4.2666=7.39008$

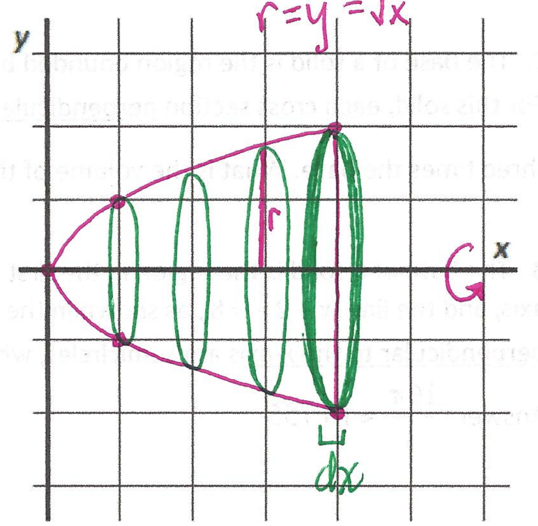
DAY 17 CLASS NOTES WED. MAR 14 PI-DAY

Part 2b: Volumes of Revolution (Disks)

Example 1: The region between the curve $y=\sqrt{x},\ 0 \le x \le 4$ and the x-axis is revolved around the x-axis to generate a solid. Find its volume.

Follow these steps to visualize the volumes and process the algebraic solution:

- Draw the bounded region described.
- Reflect the region in the x-axis.
- Draw three representative circular slices or disks perpendicular to the axis of rotation.
- Identify the radius of each disk in terms of the variable x.
- Identify the area of each disk in terms of the variable x.
- Disks perpendicular to a horizontal axis of rotation will have a dx-thickness.
- Identify the volume of each disk in terms of the variable x.
- Write an integral to sum the volume of all the disks.



Show work here:

$r = y = \sqrt{x}$

$A_1 = \pi r^2 = \pi(\sqrt{x})^2 = \pi x$

$V_1 = \pi x\, dx$

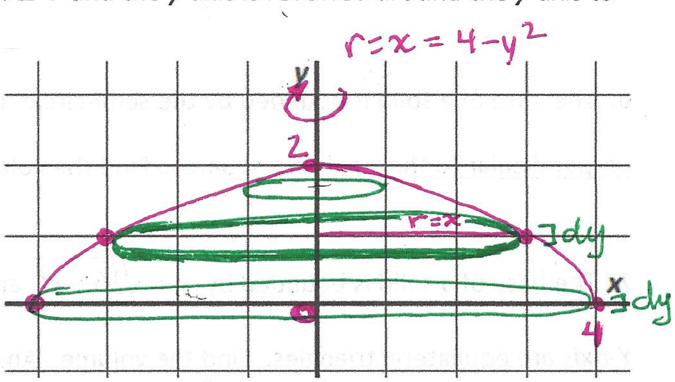
$V = \pi \int_0^4 x\, dx = \frac{\pi}{2}(x^2 \Big|_0^4 = \frac{\pi}{2}(4^2 - 0^2) = 8\pi$

Example 2: The region between the curve $y=\sqrt{4-x},\ 0 \le x \le 4$ and the y-axis is revolved around the y-axis to generate a solid. Find its volume.

$y^2 = 4 - x$

$x = 4 - y^2$

Follow the steps outlined above. Since the region is revolved about the y-axis the disks are now perpendicular to the y-axis and they have a dy-thickness.



Show work here:

$r = x = 4 - y^2$

$A_1 = \pi r^2 = \pi(4-y^2)^2$

$V_1 = \pi(4-y^2)^2\, dy$

$V = \pi \int_0^2 (4-y^2)^2\, dy = \pi \int_0^2 (16 - 8y^2 + y^4)\, dy$

$\frac{15}{15} - \frac{10}{15} + \frac{2}{15}$

$V = \pi\left(16y - \frac{8}{3}y^3 + \frac{1}{5}y^5\right)\Big|_0^2 = \pi\left(\left(32 - \frac{64}{3} + \frac{32}{5}\right) - 0\right) = 32\pi\left(1 - \frac{2}{3} + \frac{1}{5}\right) = 32\pi\left(\frac{7}{5}\right)$

$= \frac{224\pi}{5}$

Volume of Revolution by Disks

Volume of a solid obtained by revolving a bounded region about an axis of rotation:

$$V = \int_a^b \pi[R(x)]^2\, dx \quad or \quad V = \int_c^d \pi[R(y)]^2\, dy$$

$$for\ \ a \le x \le b \qquad\qquad for\ \ c \le y \le d$$

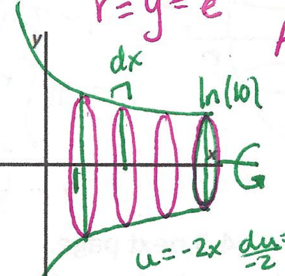
DAY 117 CLASSNOTES.

Problems: Graph the region. Write and evaluate an integral to find the volume of the solid generated by revolving the region about the stated axis. #1 & #8 – do by hand. #2-6 – evaluate with a calculator.

1. $y = e^{-x},\ y = 0,\ x = 1,\ x = \ln(10)$

$x-axis$

$r = y = e^{-x}$ $A = \pi\left(e^{-2x}\right)$



#1 ANS $\frac{\pi}{2}\left(\frac{1}{e^2} - \frac{1}{100}\right)$

$u = -2x \quad \frac{du}{-2} = dx$

$$V = \pi\int_1^{\ln 10} e^{-2x}\,dx = \frac{\pi}{-2}\int_{-2}^{-2\ln 10} e^u\,du$$

$$V = -\frac{\pi}{2} e^u\Big|_{-2}^{-2\ln 10} = -\frac{\pi}{2}\left(e^{-2\ln 10} - e^{-2}\right)$$

$$V = \frac{\pi}{2}\left(e^{-2} - e^{-2\ln 10}\right) = \frac{\pi}{2}\left(\frac{1}{e^2} - \frac{1}{100}\right)$$

#2 ANS 53.6165 $\frac{256\pi}{15}$

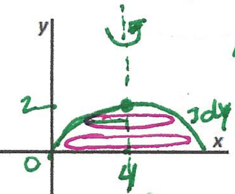
2. $y = \sqrt{x},\ y = 0,\ x = 4$

$axis: x = 4$

$x = y^2$

$r = 4 - \sqrt{x} = 4 - y^2$

$A = \pi\left(4 - y^2\right)^2$



$$V = \pi\int_0^2 \left(4 - y^2\right)^2 dy = V = \pi\int_0^2 \left(16 - 8y^2 + y^4\right)dy$$

$$V = \pi\left(16y - \frac{8}{3}y^3 + \frac{1}{5}y^5\right)\Big|_0^2$$

$$V = \pi\cdot\left(32 - \frac{64}{3} + \frac{32}{5}\right) = 32\pi\left(1 - \frac{2}{3} + \frac{1}{5}\right)$$

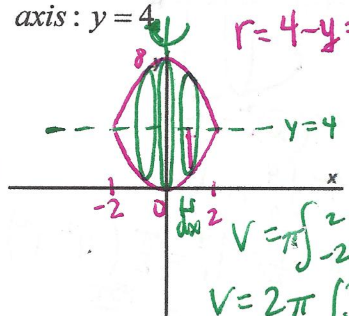
$$V = 32\pi\left(\frac{15 - 10 + 3}{15}\right) = \frac{256\pi}{15}$$

3. $y = x^2,\ y = 4$

$axis: y = 4$

$r = 4 - y = (4 - x^2)$

$A = \pi\left(4 - x^2\right)^2$



#3 ANS 107.2330 $\frac{512\pi}{15}$

#4 ANS 188.4955 60π

$$V = \pi\int_{-2}^{2} \left(4 - x^2\right)^2 dx$$

$$V = 2\pi\int_0^2 \left(4 - x^2\right)^2 dx$$

$$V = 2\pi\int_0^2 \left(16 - 8x^2 + x^4\right) dx$$

$$V = 2\pi\left(16x - \frac{8}{3}x^3 + \frac{1}{5}x^5\right)\Big|_0^2$$

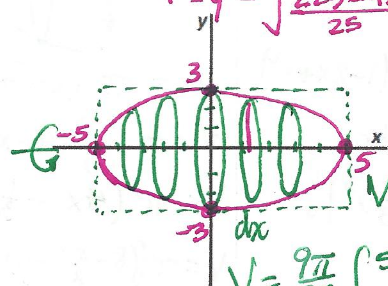
$$V = 2\pi\left(32 - \frac{64}{3} + \frac{32}{5}\right) = 64\pi\left(1 - \frac{2}{3} + \frac{1}{5}\right)$$

$$64\pi\left(\frac{15 - 10 + 3}{15}\right) = \frac{512\pi}{15}$$

4. $9x^2 + 25y^2 = 225$ (ellipse) $\frac{x^2}{25} + \frac{y^2}{9} = 1 \quad \left(\frac{x}{5}\right)^2 + \left(\frac{y}{3}\right)^2 = 1$

$x-axis$

$r = y = \sqrt{\frac{225 - 9x^2}{25}} = \frac{3}{5}\sqrt{25 - x^2}$



$$A = \pi\left(\frac{3}{5}\sqrt{25 - x^2}\right)^2$$

$$A_i = \frac{9\pi}{25}\left(25 - x^2\right)$$

$$V_i = \frac{9\pi}{25}\left(25 - x^2\right)dx$$

$$V = \frac{9\pi}{25}\int_{-5}^{5}\left(25 - x^2\right)dx$$

$$V = \frac{18\pi}{25}\int_0^5 \left(25 - x^2\right)dx$$

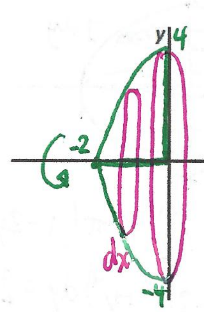
$$\frac{18\pi}{25}\left(25x - \frac{1}{3}x^3\right)\Big|_0^5 = \pi\frac{18}{25}\left(125\left(1 - \frac{1}{3}\right)\right)$$

$$= \pi\,18\cdot 5\cdot\frac{2}{3} = 60\pi = 188.495$$

5. $y = 4 - x^2,\ y = 0,\ x = -2,\ x = 0$

$x-axis$

$r = y = 4 - x^2$



#5 ANS 53.6165 $\frac{256\pi}{15}$

#6 ANS 2.467 $\frac{\pi^2}{4}$

$$A = \pi\left(4 - x^2\right)^2$$

$$A = \pi\left(16 - 8x^2 + x^4\right)$$

$$V = \pi\int_{-2}^{0}\left(4 - x^2\right)^2 dx = \underline{53.6165}$$

$$V = \pi\int_{-2}^{0}\left(16 - 8x^2 + x^4\right)dx$$

$$V = \pi\left(16x - \frac{8}{3}x^3 + \frac{1}{5}x^5\right)\Big|_{-2}^{0}$$

$$V = \pi\left(0 - \left(-32 + \frac{64}{3} - \frac{32}{5}\right)\right)$$

$$V = \pi\left(-(-32)\left(1 - \frac{2}{3} + \frac{1}{5}\right)\right)$$

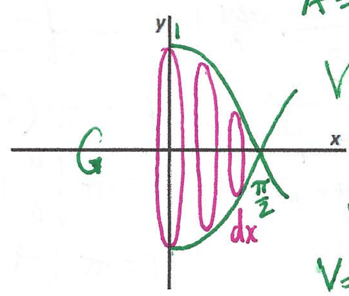
$$V = 32\pi\left(\frac{15 - 10 + 3}{15}\right) = \frac{256\pi}{15}$$

6. $y = \cos(x),\ y = 0,\ x = 0,\ x = \frac{\pi}{2}$

$x-axis$

$r = y = \cos x$

$A = \pi\left(\cos x\right)^2 = \pi\cos^2 x$



$$V = \pi\int_0^{\frac{\pi}{2}} \boxed{\cos^2 x}\,dx = \underline{2.467}$$

$$V = \pi\int_0^{\frac{\pi}{2}} \frac{1}{2}\left(1 - \cos 2x\right)dx$$

$$V = \frac{\pi}{2}\int_0^{\frac{\pi}{2}}\left(1 - \cos(2x)\right)dx$$

$$V = \frac{\pi}{2}\left(x - \frac{1}{2}\sin(2x)\right)\Big|_0^{\frac{\pi}{2}}$$

$$V = \frac{\pi}{2}\left(\left(\frac{\pi}{2} - \frac{1}{2}\sin(\pi)\right) - (0)\right)$$

$$V = \frac{\pi}{2}\cdot\frac{\pi}{2} = \frac{\pi^2}{4}$$

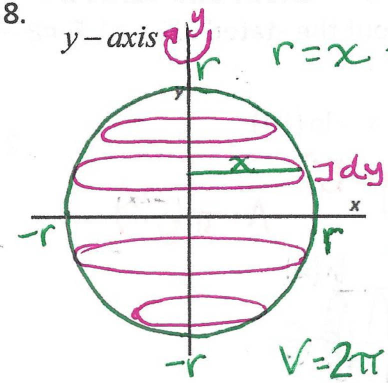
DAY 117 CLASS NOTES

7. What is the formula for the volume of a sphere?

$V = \frac{4}{3}\pi r^3$

FYI. $\frac{dV}{dr} = 4\pi r^2 = SA$

8. *a circle with radius "r"*



EQN OF CIRCLE:

$x^2 + y^2 = r^2$ (solve for x)

$r = x = \sqrt{r^2 - y^2}$

$A_1 = \pi r^2 = \pi(r^2 - y^2)$

$V_1 = \pi(r^2 - y^2)\,dy$

$V = \pi \int_{-r}^{r} (r^2 - y^2)\,dy$

$V = 2\pi\left(r^2 y - \frac{1}{3}y^3\right)\Big|_0^r = 2\pi\left(r^3 - \frac{r^3}{3}\right)$

$= 2\pi\left(\frac{2r^3}{3}\right) = \frac{4\pi r^3}{3}$

ANS

DAY 117 HW Volumes of Revolution by Disks: #1-2 below and AP FRQ #3-4-5 next page.

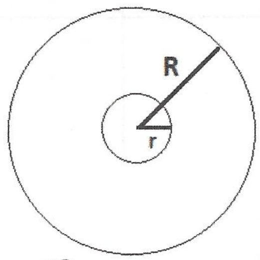
1. (Non-Calculator) Write and evaluate an integral to find the volume of the solid formed by revolving the shaded region in the X-AXIS. Show all work.

CLASSNOTES DAY 118

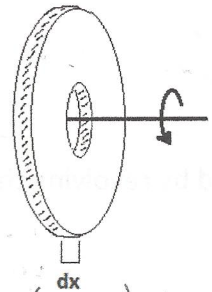
Part 2c: Volumes of Revolution (Washers)

Sometimes slicing a solid of revolution results in disks with holes in the middle—washers. In order to find these volumes, it is necessary to subtract the volume of the hole.

Area of one Washer

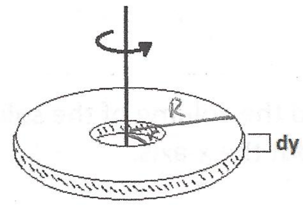


$A = \pi R^2 - \pi r^2$



$V = \pi(R^2 - r^2)\,dx$

Volume of one Washer

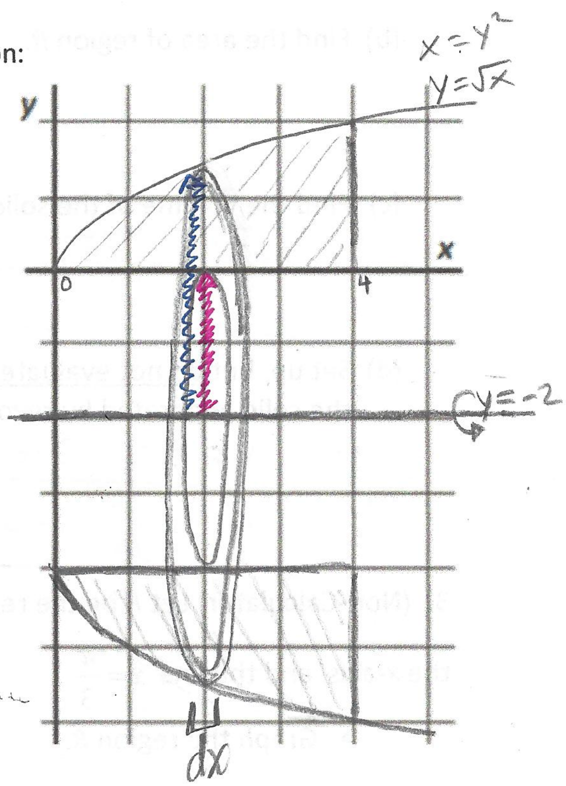


$V = \pi(R^2 - r^2)\,dy$

Example 1: The region between the curve $y = \sqrt{x},\ 0 \le x \le 4$ and the x-axis is revolved around the horizontal line $y = -2$ to generate a solid. Find its volume.

Follow these steps to visualize the volumes and process the algebraic solution:

- Draw the bounded region described. Draw the axis of rotation.
- Reflect the region in the axis of rotation.
- Draw three representative circular washer slices perpendicular to the axis of rotation.
- Identify the Outer-R and inner-r radius of each washer in terms of the variable x.
- Identify the area of each washer in terms of the variable x.
- Washers perpendicular to a horizontal axis of rotation will have a dx-thickness.
- Identify the volume of each washer in terms of the variable x.
- Write an integral to sum the volume of all the washers.



Show work here: $R = y - -2 \qquad r = 0 - -2$

$R = y + 2 \qquad r = 2$

dx-thickness $\therefore$ $R = \sqrt{x} + 2 \qquad r = 2$

$$\pi R^2 - \pi r^2 = \pi\left((\sqrt{x}+2)^2 - (2)^2\right)$$
$$= \pi\left((\sqrt{x}+2)^2 - 4\right)$$
$$= \pi(x + 4\sqrt{x} + 4 - 4) = \pi(x + 4\sqrt{x})$$

$$V = \pi\int_0^4 (x + 4\sqrt{x})\,dx$$

$$V = \pi\left(\tfrac{1}{2}x^2 + \tfrac{8}{3}\cdot x^{3/2}\right)\Big|_0^4$$

$$= \pi\left(\left(8 + \tfrac{64}{3}\right) - (0)\right) = \frac{88\pi}{3} \approx 92.153$$

CLASS NOTES DAY 118

Example 2: The region between the curve $y=\sqrt{x},\ 0\le x\le 4$ and the x-axis is revolved around the vertical line $x=-1$ to generate a solid. Find its volume.

Follow the steps outlined above. Since the region is revolved about the vertical line-axis the washers are now perpendicular to the y-axis and they have a dy-thickness. wrt y

Show work here:

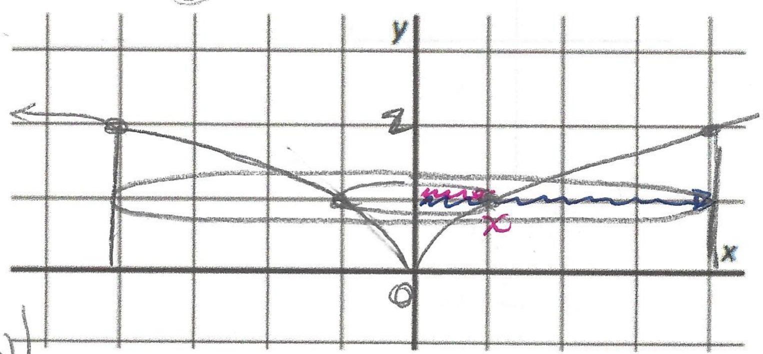
$r = x = y^2 \qquad R = 4$

$\pi\int_0^2 4^2-(y^2)^2\,dy$

$\pi\int_0^2 16-y^4\,dy$

$\pi\left(16y-\frac{1}{5}y^5\Big|_0^2\right. = \pi\left(\left(32-\frac{32}{5}\right)-(0)\right)$

$\pi\left(\frac{160-32}{5}\right) = \frac{128\pi}{5}$



Volume of Revolution by Washers **Volume of a solid obtained by revolving a bounded region about an axis of rotation:** *about a horizontal axis* *or* *about a vertical axis* $V=\pi\int_a^b\left(\left[R(x)\right]^2-\left[r(x)\right]^2\right)dx \qquad V=\pi\int_c^d\left(\left[R(y)\right]^2-\left[r(y)\right]^2\right)dy$ *for* $a\le x\le b$ *for* $c\le y\le d$

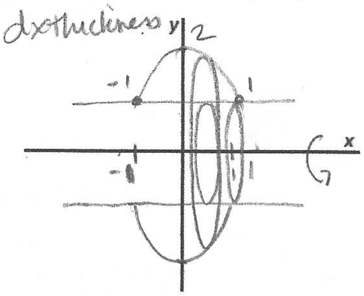
Problems:

Draw a picture and find the volume of the solid found by revolving the region about the stated axis.

ANS: $\frac{56\pi}{15}$

1. $y=2-x^2,\ y=1$ $\quad R = y = 2-x^2$
$x-axis$ $\quad r = 1$

dx thickness



$V=\pi\int_{-1}^{1}(2-x^2)^2-1^2\,dx$

$4-4x^2+x^4-1$

$\pi\int_{-1}^{1}(3-4x^2+x^4)\,dx$

$\pi\left(3x-\frac{4}{3}x^3+\frac{1}{5}x^5\Big|_{-1}^{1}\right.$

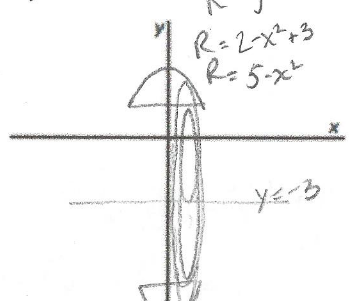
$2\pi\left(3x-\frac{4}{3}x^3+\frac{1}{5}x^5\Big|_0^1\right. = 2\pi\left(\left(3-\frac{4}{3}+\frac{1}{5}\right)-0\right)$

$2\pi\left(\frac{45-20+3}{15}\right)$ $\quad \frac{56\pi}{15}$

ANS: $\frac{176\pi}{15}$

2. $y=2-x^2,\ y=1$
$y=-3$ $\quad R = y+3 \quad r = 4$

$R = 2-x^2+3$
$R = 5-x^2$



$\pi\int_{-1}^{1}(5-x^2)^2-16\,dx$

$(25-10x^2+x^4-16)\,dx$

$\pi\int_{-1}^{1}(9-10x^2+x^4)\,dx$

$\pi\left(9x-\frac{10}{3}x^3+\frac{1}{5}x^5\Big|_{-1}^{1}\right.$

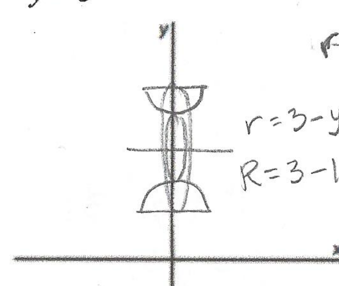
$2\pi\left(9x-\frac{10}{3}x^3+\frac{1}{5}x^5\Big|_0^1\right.$

$2\pi\left(\left(9-\frac{10}{3}+\frac{1}{5}\right)-0\right) = 2\pi\left(\frac{135-50+3}{15}\right) = \frac{176\pi}{15}$

ANS: $\frac{64\pi}{15}$

3. $y=2-x^2,\ y=1$
$y=3$

$r = 1+x^2$
$3-2+x^2$
$r = 3-y = 3-(2-x^2$
$R = 3-1 = 2$



$\pi\int_{-1}^{1} 4-(1+x^2)^2\,dx$

$4-(1+2x^2+x^4)\,dx$

$2\pi\int_0^1(3-2x^2-x^4)\,dx$

$2\pi\left(3x-\frac{2}{3}x^3-\frac{1}{5}x^5\Big|_0^1\right.$

$2\pi\left(\left(3-\frac{2}{3}-\frac{1}{5}\right)-0\right)$

$2\pi\left(\frac{45-10-3}{15}\right) = \frac{64\pi}{15}$

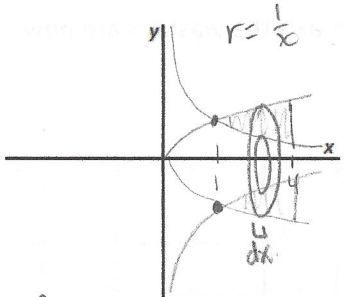
CLASS NOTES DAY 118

4. $y=\frac{1}{x},\ y=\sqrt{x},\ x=1,\ x=4$

$x-axis$

$R=\sqrt{x}$

$r=\frac{1}{x}$



$\pi\int_1^4 (\sqrt{x})^2 - \frac{1}{x^2}\,dx$

$\pi\int_1^4 x - \frac{1}{x^2}\,dx$

$\pi\left(\frac{1}{2}x^2 + \frac{1}{x}\right|_1^4$

$\pi\left(8+\frac{1}{4}\right) - \left(\frac{1}{2}+1\right)$

$\pi\left(7-\frac{1}{4}\right) = \pi\frac{27}{4} = \frac{27\pi}{4} \approx 21.2057$

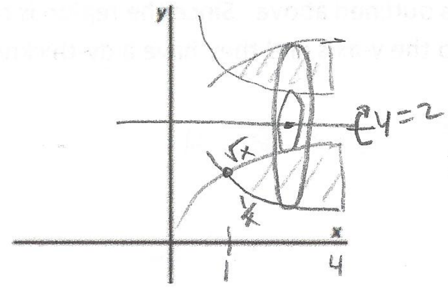
21.206

5. $y=\frac{1}{x},\ y=\sqrt{x},\ x=1,\ x=4$

$y=2$

$r=2-y=2-\sqrt{x}$

$R=2-y=2-\frac{1}{x}$



$\pi\int_1^4 \left(2-\frac{1}{x}\right)^2 - \left(2-\sqrt{x}\right)^2 dx$

$\left(4-\frac{4}{x}+\frac{1}{x^2}\right) - \left(4-4\sqrt{x}+x\right)$

$\pi\int_1^4 4\sqrt{x} - x - \frac{4}{x} + \frac{1}{x^2}\,dx$

$\pi\left(\frac{8}{3}x^{3/2} - \frac{1}{2}x^2 - 4\ln|x| - \frac{1}{x}\right|_1^4$

$\pi\left(\left(\frac{64}{3} - 8 - 4\ln|4| - \frac{1}{4}\right) - \left(\frac{8}{3} - \frac{1}{2} - 0 - 1\right)\right)$

$\pi\left(\frac{56}{3} - 7 + \frac{1}{4} - 4\ln 4\right)$

$\pi\left(\frac{143}{12} - 4\ln 4\right)$

20.0166

20.017

6. The region in the first quadrant bounded by $y=\sqrt{6x+4}$ & $y=2x$ when revolved about the $x-axis$.

$\sqrt{2(3x+2)}$ $\quad y=2x$

$\pi\int_0^2 \left(\sqrt{6x+4}\right)^2 - (2x)^2\,dx$

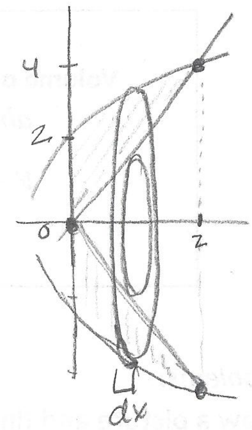
$\pi\int_0^2 6x+4-4x^2\,dx$

$\pi\left(3x^2+4x-\frac{4}{3}x^3\right|_0^2$

$\pi\left(\left(12+8-\frac{32}{3}\right)-(0)\right)$

$\frac{28\pi}{3} \approx 29.3215$

29.322

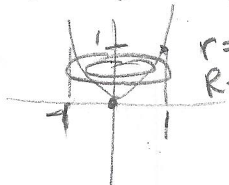


7. The region bounded by $y=x^3,\ y=0\ \&\ x=1$ when revolved about

a) the $y-axis$ dy

$r=x=\sqrt[3]{y}$

$R=1$



$\pi\int_0^1 1^2-(\sqrt[3]{y})^2\,dy$

$\pi\int_0^1 (1-y^{2/3})\,dy$

$\pi\left(y-\frac{3}{5}y^{5/3}\right)\Big|_0^1$

$\pi\left(\left(1-\frac{3}{5}\right)-(0)\right)$

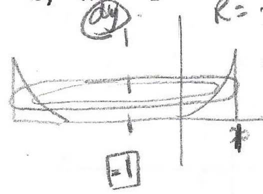
$\frac{2\pi}{5} \approx 1.2566$

1.257

b) $x=-1$ dy

$r=x+1=\sqrt[3]{y}+1$

$R=x+1=1+1=2$



$\pi\int_0^1 2^2-(\sqrt[3]{y}+1)^2\,dy$

$\pi\int_0^1 4-\left(y^{2/3}+2y^{1/3}+1\right)dy$

$\pi\int_0^1 \left(3-y^{2/3}-2y^{1/3}\right)dy$

$\pi\left(3y-\frac{3}{5}y^{5/3}-\frac{3}{2}y^{4/3}\right|_0^1$

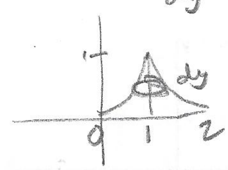
$\pi\left(\left(3-\frac{3}{5}-\frac{3}{2}\right)-(0)\right)$

$\pi\left(\frac{30-6-15}{10}\right) = \frac{9\pi}{10}$

c) $x=1$ dy

$R=1-x$

$R=1-\sqrt[3]{y}$



$\pi\int_0^1 (1-\sqrt[3]{y})^2\,dy$

$\pi\int_0^1 \left(1-2y^{1/3}+y^{2/3}\right)dy$

$\pi\left(y-\frac{3}{2}y^{4/3}+\frac{3}{5}y^{5/3}\right|_0^1$

$\pi\left(\left(1-\frac{3}{2}+\frac{3}{5}\right)-(0)\right)$

$\pi\left(\frac{10-15+6}{10}\right)$

$\frac{\pi}{10} = 0.31415$

DAY 119 NOTES

Suppose we wish to calculate the total mass or population but the density is not constant over the region, we must divide the region into smaller pieces in such a way that the density is approximately constant on each piece.

To calculate the total mass or total population where the density varies over the region,

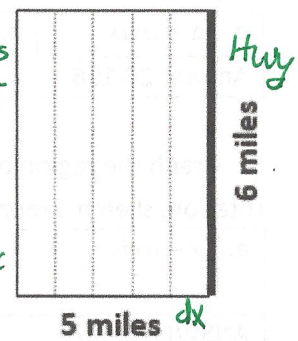
- divide the region into small pieces of relatively constant density
- add up the contributions of all the pieces
- always use units in the setups.

Example: Quadville is a city in the shape of a rectangle, five miles on one side and six miles on the other. A highway runs along the side that is six miles long. The population density x miles from the highway is given by $\boxed{p(x) = 20 - 4x}$ thousand people per square mile. What is the approximate population of Quadville?

$\boxed{\frac{1000 \text{ people}}{\text{mile}^2}}$

Solution:

1. Dimensional Analysis: What are the units involved in the scenario and what are the units of the answer? $\left(\frac{\text{thousand people}}{\text{square mile}}\right) \cdot (\text{square mile}) = \text{thousands of people}$



2. Draw a diagram including the "slices". Area $= (6)dx$

3. Determine the population of each representative slice: Population of slice $= (20-4x)(6)\,dx$

4. Determine the total population: Total $= \int_0^5 (20-4x)(6)\,dx$

Total $= 24\int_0^5 (5-x)\,dx = 24\left(5x - \frac{1}{2}x^2\right)\Big|_0^5 = 24\left(25 - \frac{25}{2}\right) = (12)(25)$

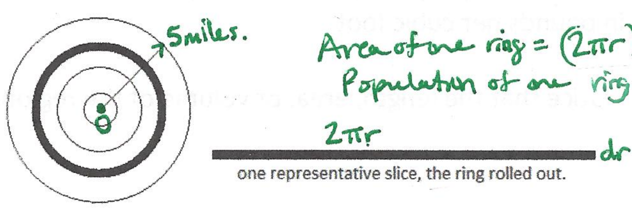
$= 300$ thousand people or 300,000 people

Ex 1: The population density in Ringsburg is a function of the distance from the city center. Suppose that at r miles from the center, the density is given by the function $p(r) = \sqrt{36 - r^2}$ in thousand persons/mile^2. Ringsburg has a radius of 5 miles. Estimate the total population of Ringsburg. Total Pop $= -\pi \int_{36}^{11} \sqrt{u}\,du$

$u = 36 - r^2$
$du = -2r\,dr$

$= \pi \int_{11}^{36} \sqrt{u}\,du = \frac{2\pi}{3}\left(u^{3/2}\Big|_{11}^{36}\right)$

$\frac{2\pi}{3}\left(6^3 - 11^{3/2}\right) = 375.97979$ thousand people



one representative slice, the ring rolled out.

Area of one ring $= (2\pi r)(dr)$

Population of one ring $= (\sqrt{36-r^2})(2\pi r)\,dr$

Total Population $= 2\pi \int_0^5 r\sqrt{36-r^2}\,dr$
$= 375.97979$

$-\frac{2}{3}\pi \cdot (36-r^2)^{3/2}\Big|_0^5 = -\frac{2}{3}\pi\left(11^{3/2} - 6^3\right)$

Ex 2: The air density (in kg/m^3) h meters above the earth's surface is $P = f(h)$. Find the mass of a cylindrical column of air 12 meters in diameter and 25 kilometers high if $f(h) = 1.28e^{-0.000124h}$. Write the Riemann sum and definite integral and then evaluate. (Hint: Be careful with units.)

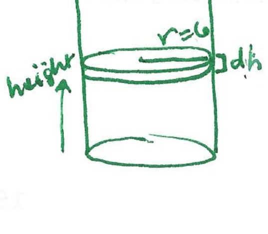
kg

25,000 meters

Dimensional Analysis:

(Air density)(Volume) = Mass

$\left(\frac{kg}{m^3}\right) \cdot (m^3) = kg$



Volume of "slice"

$\pi(6)^2\,dh$

$36\pi\,dh$

$36\pi \int_0^{25000} \left(1.28e^{-.000124h}\right) dh$

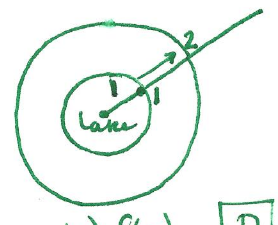
$\approx 1{,}114{,}863.389$ kg

Ex 3: A city is built around a circular lake that has a radius of 1 mile. The population density of the city is $f(r)$ people per square mile, where r is the distance from the center of the lake, in miles. Which of the following expressions gives the number of people who live within 1 mile of the lake?

(A) $2\pi \int_0^1 r f(r)\, dr$ (B) $2\pi \int_0^1 r(1+f(r))\, dr$

(C) $2\pi \int_0^2 r(1+f(r))\, dr$ (D) $2\pi \int_1^2 r f(r)\, dr$

(E) $2\pi \int_1^2 r(1+f(r))\, dr$



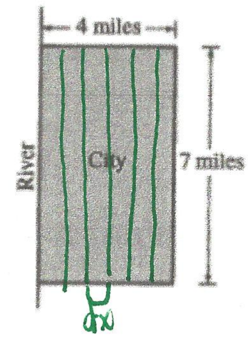
$\int_1^2 (2\pi r\, dr) f(r)$ $\boxed{D}$

$2\pi \int_1^2 r f(r)\, dr$

Ex 4: A city located beside a river has a rectangular boundary as shown in the figure above. The population density of the city at any point along a strip x miles from the river's edge is $f(x)$ people per square mile. Which of the following expressions gives the population of the city?

(A) $\int_0^4 f(x)\, dx$ (B) $7\int_0^4 f(x)\, dx$ (C) $28\int_0^4 f(x)\, dx$

(D) $\int_0^7 f(x)\, dx$ (E) $4\int_0^7 f(x)\, dx$



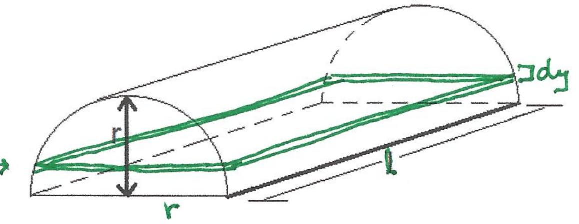
Area $= 7dx$

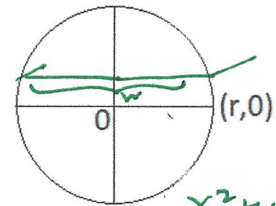
Volume $= f(x)\cdot 7dx$

$7\int_0^4 f(x)\, dx$ $\boxed{E}$

Page 447 #19 A semi-cylindrical storage shed is shown below has radius r and length l.

Slice of sawdust.





$x^2+y^2=r^2$

$x=\sqrt{r^2-y^2}$

a. What is the volume of the shed? $\left(\frac{1}{2}\pi r^2\right) l$

$V=\left(\frac{\pi}{2}r^2\right)(l)$

Width $=2x$

$w=(2\sqrt{r^2-y^2})$

$l=l$

$y=$ height off the floor

$y \in [0,r]$

b. The shed is filled with sawdust whose density (mass/unit volume) at any point is proportional to the distance of that point from the floor. The constant of proportionality is k. Calculate the total mass of the sawdust. density is proportional to distance from floor $d=ky$

mass $=$ (density)(volume) $=(ky)$

Volume of slice of sawdust

$=(2\sqrt{r^2-y^2})(l)$

mass $=\int_0^r (ky)(2\sqrt{r^2-y^2})(l)\,dy$

$2kl\int_0^r \left(y\sqrt{r^2-y^2}\right)dy$

$u=r^2-y^2$

$du=-2y\,dy$

$-\frac{1}{2}du=y\,dy$

$-kl\int_{r^2}^0 \sqrt{u}\,du$

$=kl\int_0^{r^2}\sqrt{u}\,du = kl\frac{2}{3}\left(u^{3/2}\right)\Big|_0^{r^2} = \frac{2kl}{3}(r^3-0) = \frac{2klr^3}{3}$