ANSWER KEY

§3.7 Implicit Differentiation -- A Brief Introduction -- Student Notes

Find these derivatives of these functions:

$y=\tan(x)$ $\qquad$ $y=\sin(x)$ $\qquad$ $y=e^x$

$\frac{d}{dx}(\tan(x)) = \sec^2(x)$ $\qquad$ $\frac{d}{dx}(\sin(x)) = \cos(x)$ $\qquad$ $\frac{d}{dx}(e^x) = e^x$

Write the inverses of these functions:

$y=\tan(x)$ $\qquad$ $y=\sin(x)$ $\qquad$ $y=e^x$

$x=\arctan(y)$ $\qquad$ $x=\arcsin(y)$ $\qquad$ $x=\ln(y)$

How would we find the derivatives of these inverse functions? → Good Question . . .

$\frac{d}{dx}(\underline{\arctan(y)})$ $\qquad$ $\frac{d}{dx}(\underline{\arcsin(y)})$ $\qquad$ $\frac{d}{dx}(\underline{\ln(y)})$

Let's look at a brief introduction to Implicit Differentiation so that we can find the derivatives of these three inverse functions.

Up to now, we have worked explicitly, solving an equation for one variable y in terms of another variable x. For example, if you were asked to find $\frac{dy}{dx}$ for $2x^2+y^2=4$, you would solve for y and get $y=\pm\sqrt{4-2x^2}$ and then take the derivative. This derivative requires the use of the chain rule.

Sometimes it is inconvenient, difficult or impossible to solve for y. In this case, we use implicit differentiation. It is imperative to note that anytime you see a y-variable you must think of y as a function of x just as in the notation: $y=f(x)$. Since I do not know the explicit form of $f(x)$ I will apply the chain rule to indicate it's derivative.

Differentiating with respect to x:

$\frac{d}{dx}\left[x^3\right]=3x^2$

Variables agree ⇒ use power rule

variables agree

$\frac{d}{dx}\left[y^3\right]=\left(3y^2\right)\left(\frac{dy}{dx}\right)$

Variables disagree ⇒ use power rule and chain rule

variables disagree

Since y is a function of x, we must use the chain rule & multiply by $\left(\frac{dy}{dx}\right)$.

$\frac{d}{dx}\left[2x^5+3y\right]=10x+3\left(\frac{dy}{dx}\right)$

variables disagree

variables agree

ANSWER KEY

Practice: Find $\frac{dy}{dx}$

1. $\frac{d}{dx}\left(-3y^2 = x^4 + 5\right)$

$-6y\left(\frac{dy}{dx}\right) = 4x^3$

$\frac{dy}{dx} = \frac{4x^3}{-6y} = \frac{-2x^3}{3y}$

2. $\frac{d}{dx}\left(y^2 - 7y = \cos\left(x^3\right)\right)$

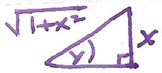
$2y\left(\frac{dy}{dx}\right) - 7\left(\frac{dy}{dx}\right) = -\sin(x^3)\cdot 3x^2$

$\frac{dy}{dx}\left[2y-7\right] = -3x^2\sin(x^3)$

$\frac{dy}{dx} = \frac{-3x^2\sin(x^3)}{2y-7}$

Let's return to our 3 inverse functions and use implicit differentiation to find their derivatives:

$\frac{d}{dx}(\arctan(x))$



$\frac{d}{dx}\left(y = \arctan(x)\right)$

$\frac{d}{dx}\left(\tan(y) = x\right)$

$\sec^2(y)\cdot\frac{dy}{dx} = 1$

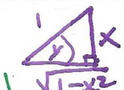
$\frac{dy}{dx} = \frac{1}{\sec^2(y)}$

$\frac{dy}{dx} = \cos^2(y)$

$\frac{dy}{dx} = \left(\frac{1}{\sqrt{1+x^2}}\right)^2$

$\frac{dy}{dx} = \frac{1}{1+x^2}$

$\frac{d}{dx}(\arcsin(x))$



$\frac{d}{dx}\left(y = \arcsin(x)\right)$

$\frac{d}{dx}\left(\sin(y) = x\right)$

$\cos(y)\frac{dy}{dx} = 1$

$\frac{dy}{dx} = \frac{1}{\cos(y)}$

$\frac{dy}{dx} = \sec(y)$

$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$

$\frac{d}{dx}(\ln(x))$

$e^{\ln x} = x$

$\frac{d}{dx}\left(y = \ln(x)\right)$

$\frac{d}{dx}\left(e^y = x\right)$

$e^y\frac{dy}{dx} = 1$

$\frac{dy}{dx} = \frac{1}{e^y}$

$\frac{dy}{dx} = \frac{1}{x}$

Derivatives of some important Inverse functions (MEMORIZE THESE).

$$\frac{d(\arctan x)}{dx} = \frac{1}{1+x^2} \qquad \frac{d(\arcsin x)}{dx} = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d(\ln x)}{dx} = \frac{1}{x}$$

* New Rules To Know!

Note: $\arctan x = \tan^{-1} x$ & $\arctan(\tan x) = \tan(\arctan x) = x$

$\arcsin x = \sin^{-1} x$ & $\arcsin(\sin x) = \sin(\arcsin x) = x$

Practice: Examples using the derivative rules we just found and applying rules we already learned: CHAIN RULE

a) $\frac{d(\arctan(t^2))}{dt}$

$= \frac{1}{1+(t^2)^2}\cdot(2t)$

$= \frac{2t}{1+t^4}$

b) $\frac{d(\arcsin(\tan(\theta)))}{d\theta}$

$= \frac{1}{\sqrt{(1-\tan^2\theta)}}\cdot\sec^2\theta$

$= \frac{\sec^2\theta}{\sqrt{1-\tan^2\theta}}$

c) $\frac{d\ln(x^2+1)}{dx}$

$= \frac{1}{(x^2+1)}\cdot 2x$

$= \frac{2x}{(x^2+1)}$

d) $\frac{d(t^2\ln t)}{dt}$ PRODUCT RULE

$= 2t\cdot\ln(t) + t^2\cdot\frac{1}{t}$

$= 2t\ln(t) + t$

$= t\cdot(2\cdot\ln(t)+1)$

$= t\cdot(\ln(t^2)+1)$

3) $\frac{d(\sqrt{1+\ln(2y)})}{dy}$

$= \frac{1}{\cancel{2}\sqrt{1+\ln(2y)}}\cdot\left(\frac{1}{2y}\right)\cancel{(2)}$

$\frac{1}{2y\sqrt{1+\ln(2y)}}$

f) $\frac{d(\cos(\sin^{-1} x))}{dx}$

$= -\underbrace{\sin\left(\sin^{-1}(x)\right)}\cdot\left(\frac{1}{\sqrt{1-x^2}}\right)$

$= -x\left(\frac{1}{\sqrt{1-x^2}}\right)$

$= \frac{-x}{\sqrt{1-x^2}}$