ANSWER KEY

NOTES

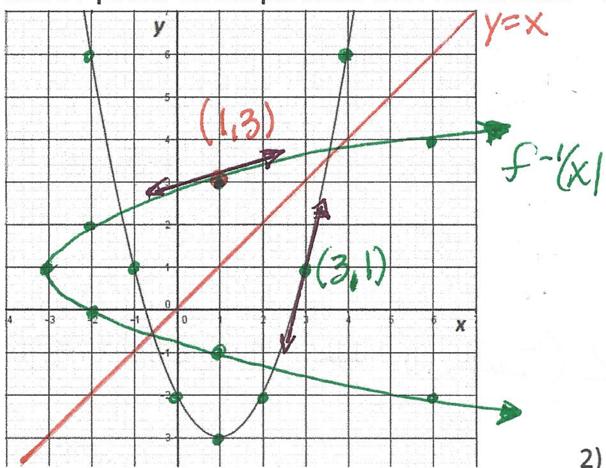
Derivative of Inverse Function Theorem

Function and Inverse Pre-requisites:

Given each function, identify key points on the function that fall on lattice points of the coordinate grid.

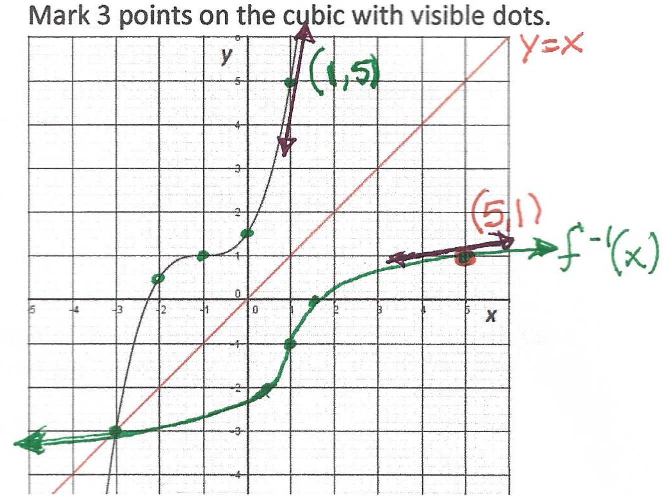
LINE OF REFLECTION.

1) Mark 7 points on the parabola with visible dots.



LINE OF REFLECTION.

2) Mark 3 points on the cubic with visible dots.



a) Write the equation of each function in (h,k) form and evaluate the function at the given point.

V(1,-3)
a=1

Quadratic function $f(x) = 1(x-1)^2-3$	$(3, f(3)) =$ $(3,1)$	Cubic function $a=\frac{1}{2}$ $(-1,1)$ $f(x) = \frac{1}{2}(x+1)^3+1$	$(1, f(1)) =$ $(1,5)$

b) For each function, list the operations on *x* that yield y.

Parabola x: -1, SQUARE, -3 x: $+3$, $\pm\sqrt{\ }$, $+1$

Cubic x: $+1$, CUBE, $\times\frac{1}{2}$, $+1$ x: ~~$\div$~~ -1, $\times 2$, $\sqrt[3]{\ }$, -1

INVERSE

c) Write inverse equations by using the list in (b) & applying *inverse operations in reverse order on x*.
State the corresponding inverse coordinate from the point on the function in part (a)

$f^{-1}(x) = \pm\sqrt{(x+3)}+1$	$(x, f^{-1}(x)) = (1,3)$	$f^{-1}(x) = \sqrt[3]{2(x-1)}-1$	$(x, f^{-1}(x)) = (5,1)$

d) Accurately, sketch the inverse function on the coordinate grid using the key lattice points. $\boxed{\text{Label } f^{-1}(x)}$ ✓

e) Find the derivative of the function at the specified point.
Find the derivative of its inverse at the corresponding point on the inverse.

$f'(x) = 2(x-1)$ (3,1)	$\left.\frac{dy}{dx}\right\|_{x=3} = 2(2) = 4$	$f'(x) = \frac{3}{2}(x+1)^2$ (1,5)	$\left.\frac{dy}{dx}\right\|_{x=1} = \frac{3}{2}(4) = 6$
$(f^{-1})'(x) = \frac{1}{2\sqrt{x+3}}$ (1,3)	$(f^{-1})'(\underline{1}) = \frac{1}{2\sqrt{4}} = \frac{1}{4}$	$(f^{-1})'(x) = \frac{2}{3(2(x-1))^{2/3}}$ (5,1)	$(f^{-1})'(\underline{5}) = \frac{2}{3(\sqrt[3]{8})^2} = \frac{2}{3(4)} = \frac{1}{6}$

f) What is the relationship between the derivative value of the function at the point and its inverse at the corresponding inverse point? ______________________________

§3.6 NOTES

AB Calculus – Supplement
Derivative of the Inverse of a Function

Name: ANSWER KEY

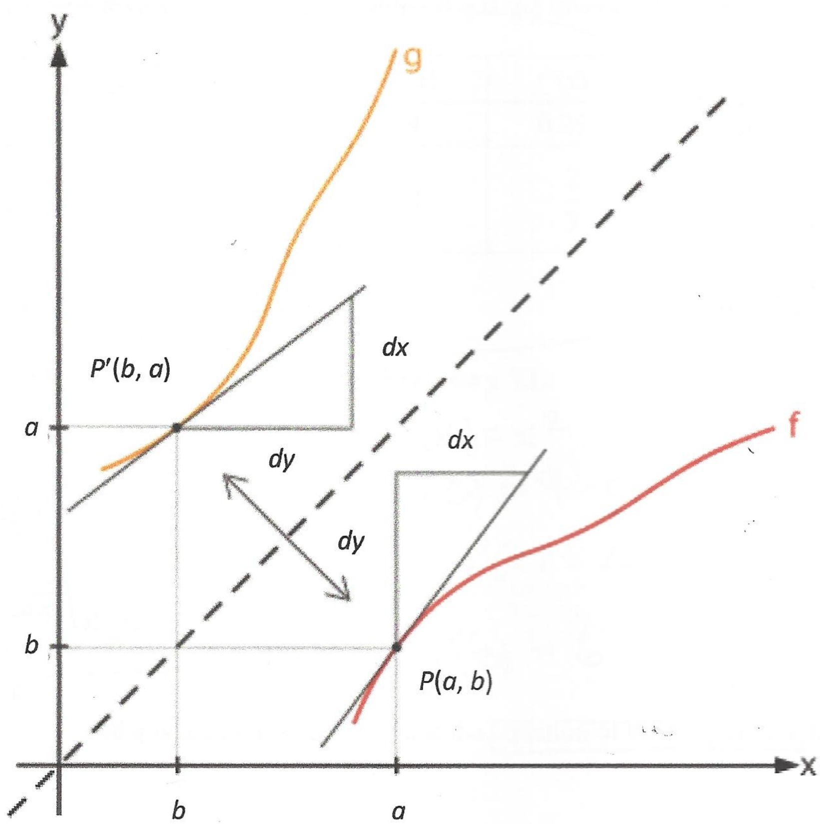
Class/Period: ________ Date: ____________

Suppose that f and g are inverse functions. What is the relationship between their derivatives?

- Algebraically: inverses are obtained by interchanging the x and the y coordinates and solving for y.
- Graphically: inverses are a reflection of the graph on the line $y = x$.

If f passes through the point (a, b), then the slope of the curve at $x = a$ is represented by $f'(a)$ and is represented by the ratio of the change in y over the change in x, $\frac{\Delta y}{\Delta x}$.

When this figure is reflected on the line $y = x$, we obtain the graph of the inverse f^{-1} and this passes through the point (b, a), with the horizontal and vertical sides of the slope triangle interchanged. So the slope of the line tangent to the graph of f^{-1} at $x = b$ is represented by the change in x over the change in y, $\frac{\Delta x}{\Delta y}$. This is the reciprocal of the slope of f at $x = a$.



http://demo.activemath.org/ActiveMath2/LeAM_calculusPics/DerivInverseFunction.png?lang=en

Given (a, b) is a point on f, and g is the inverse of f, ∴ Point on $g(x)$ (b,a)

If $f'(a) = m$, then $g'(b) = \frac{1}{m}$.

The derivative of the inverse of a function at a point is the reciprocal of the derivative of the function at the corresponding point.

§3.6 Derivative of Inverse Function ANSWER KEY NOTES.

Examples: DECODE

1) If $f(7) = 1$ and $f'(7) = 5$, and g is the inverse of f, then what is $g'(1)$?

RECIPROCALS

$\frac{8}{7} \leftrightarrow \frac{7}{8}$

$f(x): (7,1) \longrightarrow g(x): (1,7)$

$f'(7) = 5$ $\quad g'(x) = \frac{1}{5}$

2) Given $f(-2) = 5$, $\underline{f'(-2) = 6}$, ~~$f'(5) = -3$~~ and g is the inverse of f, what is $g'(5)$?

$f(x): (-2,5) \longrightarrow g(x): (5,-2)$

$f'(-2) = 6$ $\quad g'(5) = \frac{1}{6}$

3) A function f and its derivative are shown on the table. If g is the inverse of f, find $g'(4)$ and $g'(-1)$. Reciprocal

x	$f(x)$	$f'(x)$
-3	4	0.25
2	-1	$-\frac{2}{3}$

$f(x): (-3,4)$ $\quad g(x): (4,-3)$

$f'(-3) = \frac{1}{4}$ $\quad g'(4) = \frac{1}{\frac{1}{4}} = 4$

$f(x): (2,-1)$ $\quad g(x): (-1,2)$

$f'(2) = -\frac{2}{3}$ $\quad g'(-1) = -\frac{3}{2}$

4) Let $f(x) = \sqrt{x}$, and let g be the inverse function. Evaluate $g'(3)$.

$f(x) = \sqrt{x} \xrightarrow{\text{INVERSES}} g(x) = x^2$

$f(9) = 3 \quad f(x): (9,3) \longleftarrow g(3) = 9 \quad g(x): (3,9)$

$f'(x) = \frac{1}{2\sqrt{x}}\Big|_{x=9}$ $\quad g'(x) = 2x\Big|_{x=3}$

$f'(9) = \frac{1}{6} \xrightarrow{\text{RECIPROCAL}} g'(3) = 6$

5) If $f(2) = -3$, $f'(2) = \frac{3}{4}$, and g is the inverse of f, what is the $\boxed{\text{equation of the tangent line}}$ to $g(x)$ and $x = -3$?

$f(2) = -3$

$f(x): (2,-3) \longrightarrow g(x): (-3,2)$

$f'(2) = \frac{3}{4}$ $\quad g'(-3) = \frac{4}{3}$

RECIPROCAL

$\boxed{\text{ATQ}}$ EQN of TAN. LINE : $\boxed{y = \frac{4}{3}(x+3) + 2}$
on $g(x)$ @ $x = -3$

§3.6 Derivative of Inverse Function ANSWER KEY $\boxed{\text{HW DAY 55}}$

#6

6) The following figure shows $f(x)$ and $f^{-1}(x)$. Using the given table, find:

a) $f(2), f^{-1}(2), f'(2), (f^{-1})'(2)$.

$f(2)=4$ $f^{-1}(2)=1$

$f'(2)=2.8=\frac{14}{5}$ $(f^{-1})'(2)=\frac{1}{f'(1)}=\frac{1}{1.4}=\frac{5}{7}$

b) The equation of the tangent line at the points $P(3, 8)$ and $Q(8, 3)$.

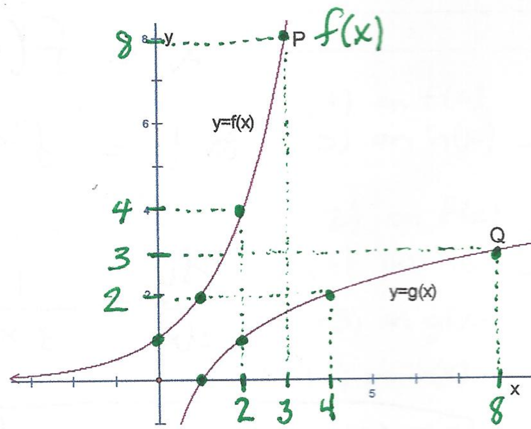
$f'(3)=5.5$ $P(3,8)$ $(f^{-1})'(8)=\frac{1}{f'(3)}=\frac{1}{5.5}=\frac{2}{11}$

$y=\frac{11}{2}(x-3)+8$ $y=\frac{2}{11}(x-8)+3$

c) What is the relationship between the two tangent lines?

c) $\boxed{\text{These tangent lines are inverses.}}$

x	$f(x)$	$f'(x)=g(x)$	
0	1	0.7	
1	2	1.4	$\to \frac{14}{10}=\frac{7}{5}$
2	4	2.8	
3	8	5.5	$\to \frac{55}{10}=\frac{11}{2}$



$f(x)=2^x$

$f'(x)=(\ln 2)2^x\big|_{x=3}$

$f'(3)=2^3\cdot\ln 2=8\ln(2)$

$g(x)=f^{-1}(x)=\log_2(x)$

$g(x)=f^{-1}(x)=\frac{\ln(x)}{\ln(2)}$

$g'(x)=(f^{-1})'(x)=\frac{1}{x\cdot\ln(2)}\Big|_{x=8}$

$\boxed{g'(2)=\frac{1}{8\ln 2}}$

7) Calculate $g'(1)$, where $g(x)$ is the inverse of the function $f(x)=x+e^x$ without solving for $g(x)$.

Ⓐ $f(x)=x+e^x=1 \longrightarrow g(1)=0$ Ⓑ

$x=0$ Ⓒ $f'(x)=1+e^x$

$\therefore f(0)=1$ $f'(0)=1+1=2 \xrightarrow{\text{reciprocal}}$ Ⓓ $g'(1)=\frac{1}{2}$

8) Calculate $g'(x)$, where $g(x)$ is the inverse of the function $f(x)=x^3+1$ without solving for $g(x)$.

$f(x)=x^3+1 \longrightarrow g(x)=f^{-1}(x)$

$f'(x)=3x^2 \xrightarrow{\text{RECIPROCAL}} g'(x)=\frac{1}{3x^2}$

9) Let $f(x)=\frac{1}{4}x^3+x-1$. Assume that $f(x)$ is one-to-one. ← meaning $f(x)$ has an inverse that is a function.

a. What is the value of $f^{-1}(x)$ when $x=3$?

$f^{-1}(3)=?$ $\therefore$ Solve $f(x)=3=\frac{1}{4}x^3+x-1$ $\therefore \frac{1}{4}x^3+x-4=0$ TI $x=2$

b. Find the slope of the tangent line to the curve $y=f^{-1}(x)$ at $x=3$.

$\therefore f(2)=3$ $f^{-1}(3)=2$

$f'(x)=\frac{3}{4}x^2+1$ $(f^{-1})'(3)=\frac{1}{4}$

$f'(2)=4$

$y=\frac{1}{4}(x-3)+2$

§3.6 Derivative of Inverse Function ANSWER KEY (HW DAY 55

Keys to Properly Solving Derivative of an Inverse Problems:

- First, identify the point (a, b) on the function f using whatever information is given.
- Differentiate f.
- Take the reciprocal of the derivative of f. This is the derivative of f^{-1}.
- Evaluate the derivative of f^{-1} at the point (b, a).

Practice:

Given the following values for differentiable functions f and g.

x	f $f(x)$	f'	g $g(x)$	g'
1	2	½	-3	5
2	3	1	0	4
3	4	2	2	3
4	6	5	3	½

a. If $h(x) = f^{-1}(x)$, what is $h'(4)$? $\to f(?)=4$ $(3,4)$ on $f(x)$ $f'(3)=2$
$(4,3)$ on $h(x)$ $\to h'(4)=\frac{1}{f'(3)}=\frac{1}{2}$

b. If $h(x) = f^{-1}(x)$, what is $h'(2)$? $\to f(?)=2$ $(1,2)$ on $f(x)$ $f'(1)=\frac{1}{2}$
$(2,1)$ on $h(x)$ $\to h'(2)=\frac{1}{f'(1)}=\frac{1}{\frac{1}{2}}=2$

c. If $d(x) = g^{-1}(x)$, what is $d'(-3)$? $\to g(?)=-3$ $(1,-3)$ on $g(x)$ $g'(1)=5$
$(-3,1)$ on $h(x)$ $\to h'(-3)=\frac{1}{g'(1)}=\frac{1}{5}$

And these are not exactly on derivatives of inverses, but they are good practice nonetheless:

d. If $p(x) = g^2(x)$, what is $p'(3)$? $p'(x)=2(g(x))^1\cdot g'(x)$ $\therefore p'(3)=2(g(3))^1\cdot g'(3)$
$=2(2)(3)=12$

e. If $b = f \bullet g$ what is $b'(2)$? $b'=f'\cdot g+f\cdot g'$ $\therefore b'(2)=f'(2)g(2)+f(2)\cdot g'(2)$
$=(1)(0)+(3)(4)=0+12=12$

f. If $n(x) = f(x^3)$, what is $n'(1)$? $n'(x)=f'(x^3)(3x^2)$ $\therefore n'(1)=f'(1^3)\cdot(3(1)^2)$
$=f'(1)\cdot(3)$
$=(\frac{1}{2})(3)=\frac{3}{2}$

$g(x) = f^{-1}(x)$

$g(x) = y = f^{-1}(x)$

$x = f(y)$

$1 = f'(y) \cdot \frac{dy}{dx}$

$\frac{dy}{dx} = \frac{1}{f'(y)}$

$$\boxed{g'(x) = \frac{dy}{dx} = \frac{1}{f'(f^{-1}(x))}}$$