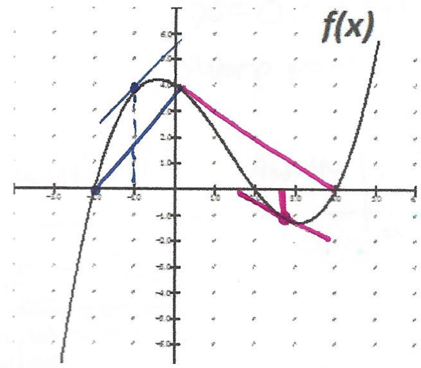
DAY 58 CLASS NOTES

§3.10 Mean Value Theorem – Student Notes

MEAN VALUE THEOREM: **If a function is <u>continuous</u> on [a, b] and <u>differentiable</u> on (a, b), then there is as number c in the interval (a, b) such that**

$$f'(c) = \frac{f(b)-f(a)}{b-a} \quad \text{or} \quad (b-a)\ f'(c) = f(b) - f(a).$$

1. Use the graph to illustrate the Mean Value Theorem with a continuous and differentiable function. Show *f (x), a, b, c* and all other conditions of the theorem.



EX. $x \in [-2, 0]$

$$\frac{f(0)-f(-2)}{0+2} = f'(-1)$$

EX. $x \in [0, 4]$

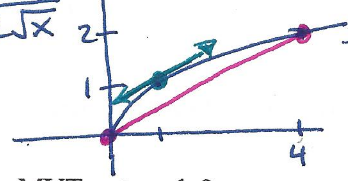
$$\frac{f(4)-f(0)}{4-0} = f'(2.8)$$

2. Find the number c that satisfies the Mean Value Theorem (MVT) for f (x) = $\sqrt{x}$ on the interval [0,4]. Draw a picture.

f(x) is continuous on the closed interval [0,4] and differentiable on the open (0,4)

$f(x) = \sqrt{x}$

$f'(x) = \frac{1}{2\sqrt{x}}$



$$\boxed{\frac{f(4)-f(0)}{4-0} = \frac{2-0}{4} = \frac{1}{2}}$$

$$\boxed{f'(1) = \frac{1}{2}}$$

$\frac{1}{2\sqrt{x}} = \frac{1}{2}$? $\rightarrow$ $\boxed{\therefore f'(1) = \frac{f(4)-f(0)}{4-0} \quad c = 1}$

3. Why does the MVT not apply?

a) $y = \frac{x+3}{x-2}$ on [0,3]

The function is not continuous at $x = 2$ $\therefore$ the MVT does not apply.

b) $f(x) = x^{\frac{1}{3}}$ on [-1,1] $\rightarrow$ $f'(x) = \frac{1}{3\sqrt[3]{x^2}}$

f(x) is continuous on [-1,1] but f(x) is not differentiable on (-1,1) b/c $f'(0)$ is undefined $\therefore$ the MVT does not apply.

4. Apply the MVT, if possible. If not possible explain why.

A $f(x) = x^2$ on [-2,1]	B $f(x) = x^3 - 3x^2$ on [0,3]	C $f(x) = x^{\frac{2}{3}}$ on [0,1]
✓ f(x) is continuous on [-2,1] ✓ f(x) is differentiable on (-2,1) $f'(x) = 2x = \frac{f(1)-f(-2)}{1+2}$ $2x = \frac{1-4}{1+2}$ $2x = \frac{-3}{3}$ $\therefore x = -\frac{1}{2}$ $f'(-\frac{1}{2}) = \frac{f(1)-f(-2)}{1+2} = -1$ $\therefore$ MVT is satisfied for f(x) on [-2,1] when $c = -\frac{1}{2}$.	✓ f(x) is continuous on [0,3] ✓ f'(x) is diff on (0,3) $f'(x) = 3x^2 - 6x$ $3x^2 - 6x = \frac{f(3)-f(0)}{3-0} = 0$ $3x(x-2) = 0$ $x = 0$ & $x = 2$ $f'(0) = f'(2) = \frac{f(3)-f(0)}{3-0} = 0$ $\therefore$ The MVT is satisfied for f(x) on [0,3] when $c = 0$ & 2.	✓ f(x) is continuous on [0,1] ✓ f'(x) is diff on (0,1) $f'(x) = \frac{2}{3}x^{-\frac{1}{3}} = \frac{2}{3\sqrt[3]{x}}$ $\frac{2}{3\sqrt[3]{x}} = \frac{f(1)-f(0)}{1-0} = 1$ $\sqrt[3]{x} = \frac{2}{3}$ $\therefore x = \frac{8}{27}$ $f'(\frac{8}{27}) = \frac{f(1)-f(0)}{1-0} = 1$ $\therefore$ MVT is satisfied for f(x) on [0,1] when $c = \frac{8}{27}$.

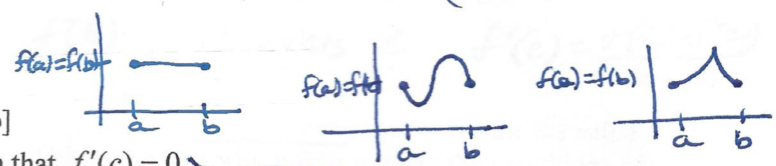
DAY 58 CLASS NOTES.

MVT Problems

1. The function $f(x)=x^{\frac{2}{3}}$ on [- 8, 8] does not satisfy the conditions of the Mean Value Theorem because
   - A. $f(0)$ is not defined
   - B. $f(x)$ is not continuous of [-8, 8]
   - C. $f'(-1)$ does not exist
   - D. $f(x)$ is not defined for x < 0.
   - E. $f'(0)$ does not exist

$f(x)$ is continuous on $[-8,8]$ but $f(x)$ is not differentiable at $x=0 \therefore f'(0)$ DNE (sharp point)

2. If $f(a) = f(b)$ and $f(x)$ is continuous on [a, b], then
   - ~~A.~~ $f(x)$ must be identically zero
   - ~~B.~~ $f'(x)$ may be different from zero for all x on [a,b]
   - ~~C.~~ there exists at least one number c, a < c< b, such that $f'(c)=0$
   - ~~D.~~ $f'(x)$ must exist for every x on (a, b)
   - E. none of the preceding is true



only if $f(x)$ is differentiable on (a,b) & MVT applies

3. Find the value of c that satisfies the Mean Value Theorem for $f(x)=x^3+x-4$ on the interval [- 2, 1].

$f(x)$ is continuous on $[-2,1]$ & differentiable on $(-2,1)$

   - A. – 1
   - B. 1
   - C. 0
   - D. 4
   - E. None of these.

$f'(c) = \frac{f(1)-f(-2)}{1+2}$

$3c^2+1 = \frac{-2+14}{3}$

$3c^2+1=4$

$3c^2=3$

$c^2=1$

$c=1$ or $c=-1$

4. Find the number that satisfies the MVT on the given interval <u>or</u> state why the theorem does not apply.

a) $f(x)=x^{\frac{2}{5}}$ on [0, 32]

Sharp pt. @ $x=0$

$f(x)$ is cont. on $[0,32]$

$f(x)$ is diff. on $(0,32)$

$f'(c) = \frac{2}{5}c^{-3/5} = \frac{2}{5c^{3/5}} = \frac{f(32)-f(0)}{32}$

$\frac{4-0}{32} = \frac{1}{8} = \frac{2}{5c^{3/5}} \rightarrow c^{3/5}=16$

$\boxed{c=16^{5/3}}$ by MVT.

b) $f(x)=\dfrac{1}{(x-2)^2}$ on [2, 5]

$f(x)$ is discontinuous at $x=2$ $\therefore$ MVT does not apply

c) $g(x)=x+\dfrac{1}{x}$ on [1, 3]

$g(x)$ is continuous on $[1,3]$ & diff on $(1,3)$ $\therefore$

$g'(c) = 1-\frac{1}{c^2} = \frac{c^2-1}{c^2} = \frac{g(3)-g(1)}{3-1}$

$\frac{c^2-1}{c^2} = \frac{\frac{4}{3}-2}{2} = \frac{2}{3}-1 = \frac{-1}{3}$

$\frac{c^2-1}{c^2} = \frac{-1}{3} \rightarrow 3c^2-3=-c^2$

$4c^2=3 \quad c=\pm\frac{\sqrt{3}}{2} \quad \therefore \boxed{c=+\frac{\sqrt{3}}{2}}$ by MVT.

d) $h(x)=x^{\frac{1}{2}}+2\underbrace{(x-2)^{\frac{1}{3}}}$ on [1, 9]

$h(x)$ is not differentiable at $x=2$ b/c $\sqrt[3]{x-2}$ has vertical tangent.

$\therefore$ MVT does not apply.

2003 #92: Let f be defined by $f(x)=x+\ln(x)$. What is the value of c for which the instantaneous rate of change of f at $x=c$ is the same as the average rate of change of f over $[1,4]$?

(A) 0.456 (B) 1.244 (C) 2.164 (D) 2.342 (E) 2.452

$f(x)=x+\ln(x)$ is continuous on $[1,4]$ & differentiable on $(1,4)$.

$f'(x) = 1+\frac{1}{x} = \frac{x+1}{x} = \frac{f(4)-f(1)}{3} =$

$1+\frac{1}{x} = \frac{x+1}{x} = \frac{(4+\ln 4)-(1+\ln 1)}{3} = \frac{3+\ln 4}{3} = \frac{x+1}{x}$

or $1+\frac{1}{x} = 1+\frac{\ln 4}{3}$

$\frac{3+\ln 4}{3} = \frac{x+1}{x}$

$(3+\ln 4)x = 3x+3$

$(3+\ln 4)x - 3x = 3$

$x(3+\ln 4-3) = 3$

$x(\ln 4) = 3$

$x = \frac{3}{\ln 4}$

"RECALL" $\boxed{\text{DAY 58}}$ ✱ HW ✓

R1 **HW MVT**
Write the definition of continuity.

1) $f(c)$ exists	2) $\lim_{x\to c} f(x)$ exists	3) $f(c) = \lim_{x\to c} f(x)$

R2 Write mathematical notation for differentiability: $f'(c^-) = f'(c^+)$ or $\lim_{x\to c^-} f'(x) = \lim_{x\to c^+} f'(x) = f'(c)$

R3 State the two prerequisite conditions that must be determined before the Mean Value Theorem can be applied.

1) $f(x)$ must be continuous on the closed interval $[a,b]$
2) $f(x)$ must be differentiable on the open interval (a,b)

R4 What two calculations must be determined before making a conclusion using the Mean Value Theorem.

1) $f'(c)$ exists
2) $\frac{f(b)-f(a)}{b-a}$ exists & $f'(c) = \frac{f(b)-f(a)}{b-a}$

HW. Read questions #1-4. If the function satisfies the hypotheses of the Mean Value Theorem, then solve for the value of c that satisfies the conclusion of the Mean Value Theorem. Otherwise, tell why it fails to meet the conditions of the Mean Value Theorem.

1. Given $f(x) = 5 - \frac{4}{x}$, find all values, c, in the interval $[1,4]$.

$f(x)$ is continuous on $[1,4]$ & differentiable on $(1,4)$
$\therefore$ MVT applies &

$\frac{f(4)-f(1)}{4-1} = \frac{4-1}{4-1} = 1 = f'(c) = \frac{4}{c^2}$

$f'(x) = \frac{4}{x^2}$ $\quad 1 = \frac{4}{c^2}$ $\quad c = \pm 2$

$\therefore$ $\boxed{\therefore \text{By MVT } c = 2,\ f'(2) = \frac{f(4)-f(1)}{4-1} = 1}$

2. Given $f(x) = x^4 - 2x^2$, find all values, c, in the interval $[-2,2]$.

f is continuous on $[-2,2]$ & differentiable on $(-2,2)$
$\therefore$ MVT applies &

$\frac{f(2)-f(-2)}{2+2} = \frac{8-8}{2+2} = \frac{0}{4} = 0$

$f'(x) = 4x^3 - 4x = 4x(x^2-1)$
$4x(x-1)(x+1) = 0 \quad x = 0, 1, -1$

$\boxed{\therefore \text{By MVT when } c = 0, -1, +1 \quad f'(c) = \frac{f(2)-f(-2)}{4} = 0}$

3. Given $f(x) = x(x^2 - x - 2)$, find all values, c, in the interval $[-1,1]$.

$f(x)$ is continuous on $[-1,1]$ and differentiable on $(-1,1)$
$\therefore$ MVT applies &

$\frac{f(1)-f(-1)}{1+1} = \frac{-2-0}{2} = -1$

$f'(x) = 3x^2 - 2x - 2$

$3x^2 - 2x - 2 = -1$
$3x^2 - 2x - 1 = 0$
$(3x+1)(x-1) = 0$
$x = -\frac{1}{3} \quad x = 1$

$\boxed{\therefore \text{By MVT when } c = -\tfrac{1}{3}, 1 \quad f'(c) = \frac{f(1)-f(-1)}{2} = -1}$

4. Given $f(x) = x^{\frac{2}{3}} - 1$, find all values, c, in the interval $[-8,8]$.

$f(x)$ is continuous on $[-8,8]$
but $f(x)$ is not differentiable on $(-8,8)$
b/c $f'(0)$ does not exist (sharp point on $f(x)$)
So MVT does not apply.