MOTION KEY p. 1

AP Calculus AB— Unit 6 Motion with Integrals:
Position – Velocity – Acceleration – Speed – Total Distance – Displacement

Motion with Integrals
Worksheet 4: What you need to know about Motion along the x-axis (Part 2)

1. Speed is the absolute value of ___velocity___.
2. If the velocity and acceleration have the ___same___ sign (either both positive or both negative), then speed is ___increasing___.
3. If the velocity and acceleration have the ___opposite___ sign (one positive and one negative), then speed is ___decreasing___.

There are three ways to use an integral in the study of motion that are easily confused. Watch out!

4. $\int v(t)\,dt$ is an ___indefinite___ integral. It will give you an expression for ___Position___ at time t. Don't forget that you will have a ___+c___ (constant of integration), the value of which can be determined if you know a position value at a particular time.

5. $\int_{t1}^{t2} v(t)\,dt$ is a ___definite___ integral and so the answer will be a ___number___. The numerical value represents the change in ___position___ over the time Interval from ___t_1___ to ___t_2___. By the Fundamental Theorem of Calculus, since $v(t)=x'(t)$, the integral will yield $\underline{x(t2)-x(t1)}$. This is also known as displacement. The answer can be positive or negative depending on if the particle lands to the ___right___ or to the ___left___ of its original ___starting___ position.

6. $\int_{t1}^{t2} |v(t)|\,dt$ is another example of a ___definite___ integral and so the answer will be a ___number___. The numerical value represents the ___total distance___ traveled by the particle over the time interval from ___t_1___ to ___t_2___. The answer should always be ___Positive or zero___.

non-negative

Motion Key p2

AP Calculus AB— Unit 6 Motion with Integrals:
Position – Velocity – Acceleration – Speed – Total Distance – Displacement

More Reasoning with Tabular Data

4. The rate at which water is being pumped into a tank is given by the continuous, increasing function, $R(t)$. A table of selected valued of $R(t)$, for the time interval $0 \le t \le 20$ minutes, is shown above.

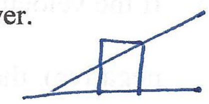
4 5 8 -3

t (min)	0	4	9	17	20
$R(t)$ (gal / min)	25	28	33	42	46

a. Use a right Riemann sum with four subintervals to approximate the value of $\int_0^{20} R(t)dt$.

Is your approximation greater of less than the true value? Give a reason for your answer.

$\int_0^{20} R(t)dt \approx 4(28) + 5(33) + 8(42) + 3(46) = 751$



751 is an overestimate b/c $R(t)$ is increasing.

b. A model for the rate at which the water is being pumped into the tank is given by the function: $W(t) = 25e^{0.03t}$, where t is measured in minutes and $W(t)$, is measures in gallons per minute. Use the model to find the average rate at which water is being pumped into the tank from time $t = 0$ to $t = 20$ minutes.

$\frac{1}{20-0}\int_0^{20} w(t)\,dt = 34.254$ gal/min

$\left(\frac{1}{\cancel{min}}\right)\left(\frac{gal}{min}\right)(\cancel{min})$

$\int_0^{20} w(t)\,dt = 685.099$ gallons

c. The tank contained 100 gallons of water at time $t = 0$ minutes. Use the model in part (b) to find the amount of water in the tank at $t = 20$ minutes.

$T(20) = T(0) + \int_0^{20} w(t)dt = 100 + 685.099 = 785.099$ gallons of water in the tank at time $t = 20$ min.

$T(t)$ = gallons of water in the Tank @ time t.

5. Car A has a positive velocity $V_A(t)$ as it travels on a straight road, where V_A is measured in (feet/sec) is a differentiable function of time t in (seconds). The velocity over the time interval $0 \le t \le 10$ seconds is shown in the table above.

2 3 2 3

t (sec)	0	2	5	7	10
$V_A(t)$ (ft / sec)	0	9	36	61	115

a. Use the data in the table to approximate the acceleration of Car A at $t = 8$ seconds. Indicate units of measure. $a(t) = \frac{v(10)-v(7)}{10-7} = \frac{115-61}{10-7} = \frac{54}{3} = 18 \frac{ft}{sec^2}$

b. Use data from the table to approximate the distance traveled by Car A over the interval $0 \le t \le 10$ seconds by using a trapezoidal sum with four subintervals. Show the computations that lead to your answer and indicate units of measure.

$\int_0^{10} |v(t)|\,dt = \frac{1}{2}(2)(0+9) + \frac{1}{2}(3)(9+36) + \frac{1}{2}(2)(36+61) + \frac{1}{2}(3)(61+115)$

$= 437.5$ ft. This value is the Total distance traveled by car A on time interval 0 to 10 seconds.

c) continued on next page ...

AP Calculus AB— Unit 6 Motion with Integrals:
Position – Velocity – Acceleration – Speed – Total Distance – Displacement

(#5 - continued)

c. Car B travels along the same road with an acceleration of $a_B(t) = 2t + 2\ (ft/\sec^2)$. At time $t = 3$ seconds, the velocity of Car B is 11 ft/sec. Which car is traveling faster at time $t = 7$ seconds? Explain your answer.

$V_B(7) = 11 + \int_3^7 (2t+2)\,dt$
$= 11 + t^2 + 2t \Big|_3^7$
$= 11 + (63 - 15) = 59$ ft/sec

$V_A(7) = 61$ ft/sec $> V_B(7) = 59$ ft/sec
$\therefore$ car A is traveling faster than car B at $t = 7$ seconds

6. A particle moves along a horizontal line with a positive velocity $v(t)$, where is measured in (cm/sec) is a differentiable function of time t in (seconds). The velocity of the particle at selected times is given in the table above.

t (sec)	0	2	4	6	8	10	12
$v(t)$ $(cm/\sec)$	37	17	5	1	6	17	38

a. Based on the values in the table, what is the smallest number of times at which the velocity pf the particle could equal 20 cm/sec in the open interval $0 < t < 12$ seconds? Justify your answer.

V(t) is a continuous function, since it is differentiable, and the Intermediate Value theorem guarantees that the particle's velocity will be 20 m/sec at least twice on the intervals
$(0,2)$ since $v(2) < 20 < v(0)$
& $(10,12)$ since $v(10) < 20 < v(12)$.

b. Based on the values in the table, what is the smallest number of times at which the acceleration of the particle could equal zero in the open interval $0 < t < 12$ seconds? Justify your answer.

· $a(t) = 0$ at least once on $(0,12)$ b/c $v(t)$ is decreasing on $(0,6)$ and increasing on $(6,12)$ $\therefore$ $a(t) = v'(t)$ must change signs from $\ominus$ to $\oplus$ for some $t \in (0,12)$. Also $\frac{v(10) - v(2)}{10 - 2} = 0$ $\therefore$ $a(t) = 0$ on $(0,12)$ between $t \in (2,10)$ by the Mean Value Theorem since $v(t)$ is continuous on $[0,12]$ & differentiable on $(0,12)$.

c. Find the average acceleration of the particle over the time interval $8 < t < 10$ seconds? Show the computations that lead to your answer and indicate units of measure.

$\frac{v(10) - v(8)}{10 - 8} = \frac{17 - 6}{2} = \frac{11}{2}\ \frac{cm}{sec^2}$

d. Use a midpoint Riemann sum with three subintervals of equal length and values from the table to approximate: $\int_0^{12} v(t)dt$. Show the computations that lead to your answer. Using correct units, explain the meaning of this definite integral in terms of the particle's motion.

$\Delta t = 4$

$\int_0^{12} v(t)\,dt = 4(17 + 1 + 17) = 140$ cm

$\frac{cm}{sec} \cdot sec$

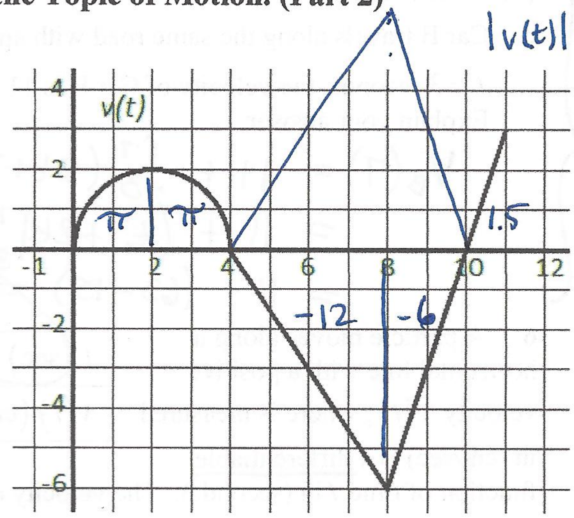
At $t = 12$ sec, the particle is 140 cm to the right of its starting position at time $t = 0$.
$\therefore$ Displacement of the particle is 140 cm to the right.

Motion Key 4.

AP Calculus AB— Unit 6 Motion with Integrals:
Position – Velocity – Acceleration – Speed – Total Distance – Displacement

Worksheet 5: Sample Practice Problems for the Topic of Motion. (Part 2)

Example 1 (graphical) No units given



The graph to the right shows the velocity $v(t)$, of a particle moving along the x-axis for $0 \le t \le 11$. It consists of a semi-circle and two line segments. Use the graph and your knowledge of motion to answer the following questions.

1. At what time t on $0 \le t \le 11$ is the speed of the particle the greatest?

   Speed = $|v(t)|$ is greatest at time $t=8$

2. At which of the times $t=2$, $t=6$, *or* $t=9$ is the acceleration of the particle the greatest? Explain your answer.

   $a(2) = v'(2) = 0$
   $a(6) = v'(6) = -1.5$
   $a(9) = v'(9) = 3$

   $a(t) = v'(t)$
   $a(t)$ is greatest at time $t=9$ when $a(9)=3$.

3. Over what time intervals is the particle moving to the left? Explain your answer.

   The particle is moving left on $(4,10)$ b/c $v(t)<0$

4. Over what time intervals is the speed of the particle decreasing? Explain your answer.

   The speed of the particle is decreasing on $t\in(2,4)$ b/c $v(t)>0$ and $v(t)$ decreasing $\therefore a(t)=v'(t)<0$ and on $t\in(8,10)$ b/c $v(t)<0$ & $v(t)$ increasing $\therefore a(t)=v'(t)>0$.

5. Find the total distance traveled by the particle over the time interval $0 \le t \le 11$.

   $\int_0^{11} |v(t)|\,dt = 2\pi - (-12-6) + 1.5 = 2\pi + 19.5$
   $2\pi + 12 + 6 + 1.5 =$

6. Find the value of $\int_0^{11} v(t)\,dt$ and explain the meaning of this integral in the context of the problem.

   $\int_0^{11} v(t)\,dt = 2\pi - 12 - 6 + 1.5 = 2\pi - 16.5 \approx -10.216$ or -10.217

   At $t=11$, the particle is $(2\pi - 16.5)$ units to the left of its original starting position when $t=0$.

7. If at time $t=0$, the particle's initial position is $x(0)=2$, complete the equation for the position of the particle at time $t=11$.

   $x(11) =$ $x(0) + \int_0^{11} v(t)\,dt$
   $= 2 + (2\pi - 16.5)$
   $= 2\pi - 14.5$

MOTION KEY p.5.

Example 2 (analytical/graphical/calculator active)

$r(t)$ $\frac{km}{hr}$

The rate of change, in kilometers per hour, of the altitude of a hot air balloon is given by $r(t)=t^3-4t^2+6$ for $0\le t\le 4$, where t is measured in hours. Assume the balloon is initially at ground level. $h(0)=0$

1. For wat values of t, $0\le t\le 4$, is the altitude of the balloon decreasing? Justify your answer.

   $r(t)<0$
   $t^3-4t^2+6<0$
   $t\in(1.5719933, 3.5141369)$
   STO▶$\boxed{A}$, STO▶$\boxed{B}$

   STORE VALUES

   The altitude of the balloon is decreasing on $t\in(1.572, 3.514)$ hrs. b/c $r(t)<0$

2. Find the values of $r'(2)$ and explain the meaning of the answer in the context of the problem. Indicate units of measure.

   $r'(t)=3t^2-8t$
   $r'(2)=12-16$
   $=-4$ km/hr^2

   At time $t=2$, the rate of change of the rate of change of the altitude of the balloon is decreasing at a rate of 4 $\frac{km}{hr^2}$
   Acceleration of the balloon is -4 km/hr^2
   or Velocity is decreasing at a rate of 4km/hr^2.

3. What is the altitude of the balloon when it is closest to the ground during the time interval, $2\le t\le 4$? The balloon will be closest to the ground at $t=3.514=\boxed{B}$

   height of balloon
   $h(B) = h(0) + \int_0^B r(t)\,dt$
   $h(3.514) = h(0) + \int_0^{3.514} r(t)\,dt = 0 + 1.34807 = 1.348$ km

4. Find the value of $\int_0^4 r(t)\,dt$ and explain the meaning of the answer in the context of the problem. Indicate units of measure.

   $\int_0^4 r(t)\,dt = 2.666$ km = Total change in altitude on (0,4) hrs

   The altitude of the balloon at $t=4$ hours is $2.66\overline{6}$ km
   Since the balloon started a ground level, the balloon at time $t=4$ hrs is $2.66\overline{6}$ km above the ground.

5. Find the value of $\int_0^4 |r(t)|\,dt$ and explain the meaning of the answer in the context of the problem. Indicate units of measure.

   $\int_0^4 |v(t)|\,dt = 11.528701 = 11.528$ km

   The balloon has travelled a total vertical distance of 11.528 km over the time interval (0,4) hrs.

6. What is the maximum altitude of the balloon during the time interval $0\le t\le 4$?

   When $r(t)$ changes signs from positive to negative at $t=1.571=\boxed{A}$
   OR at time $t=4$ (the right endpoint) the balloon may be at a maximum altitude.

   $h(A) = h(1.571) = h(0) + \int_0^{1.571=A} r(t)\,dt = 5.779$ km.
   $h(4) = h(0) + \int_0^4 r(t)\,dt = 2.666$ km

   $h(A) > h(4)$ $\therefore$ Maximum altitude of the balloon is 5.779 km. at time $t=1.571$ hrs.

MOTION KEY p.6

AP Calculus AB— Unit 6 Motion with Integrals:
Position – Velocity – Acceleration – Speed – Total Distance – Displacement

Example 3 (numerical)

2 4 4

time t	0	2	6	10
velocity $v(t)$	5	3	–1	–8
acceleration $a(t)$	0	–1	–3	–5

The table gives the values for the velocity and acceleration of a particle moving along the x-axis for selected values of time t. Both velocity and acceleration are differentiable functions of time t. The velocity is decreasing for all values of t, $0 \le t \le 10$. Use the data in the table to answer the questions that follow.

1. Is there a time t when the particle is at rest? Explain your answer.

Yes. Velocity is differentiable & therefore continuous so the Intermediate Value theorem guarantees a time when $v(t)=0$. Since $v(2)=3$ & $v(6)=-1$ there must be a value of t on $(2,6)$ when $v(t)=0$.

2. At what time indicated in the table is the speed of the particle decreasing? Explain your answer.

The speed of the particle is decreasing at time $t=2$ b/c $v(2)>0$ & $a(2)<0$.

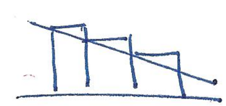
3. Use a left Riemann sum to approximate $\int_0^{10} v(t)\,dt$. Show the computations that lead to your answer. Explain the meaning of the definite integral in the context of the problem.

$\int_0^{10} v(t)\,dt = 2(5)+4(3)+4(-1) = 18$ This definite integral represents the net change in position, or displacement & means the particle is 18 units to the right of its initial position at time $t=10$.

4. Is the approximation greater or less than the actual value of the definite integral? Explain you reasoning.

Because the velocity is a decreasing function the approximation 18 is greater than the actual value.

$18 > \int_0^{10} v(t)\,dt$.



5. Approximate the value of $\int_0^{10} |v(t)|\,dt$ using a trapezoidal approximation with 3-subintervals indicated by the values in the table. Show the computations that lead to your answer. Explain the meaning of the definite integral in the context of the problem.

$\int_0^{10} |v(t)|\,dt = \frac{1}{2}(2)(5+3) + \frac{1}{2}(4)(3+1) + \frac{1}{2}(4)(+1+8) = 8+8+18 = 34$

6. Determine the value of $\int_0^{10} a(t)\,dt$. Explain the meaning of the definite integral in the context of the problem.

$\int_0^{10} a(t)\,dt = v(10) - v(0) = -8-5 = -13$ by the Fundamental Theorem of Calculus.

$\int_0^{10} a(t)\,dt = 13$ is the net change in the velocity of the particle over the time interval $(0,10)$.

MOTION Key p. 7

AP Calculus AB— Unit 6 Motion with Integrals:
Position – Velocity – Acceleration – Speed – Total Distance – Displacement

FRQ no Units given **Calculator Active**

A particle moves along the y-axis so that it's velocity, $v(t)$ at time $t \geq 0$ is given by $v(t) = 1 - \arctan(e^t)$. At time $t = 0$, the particle is at $y = -1$.

1. Find the acceleration of the particle at $t = 2$. $a(2) = \frac{d}{dt}(v(t))\Big|_{t=2} = -0.132$ or $-0.133 = a(2)$

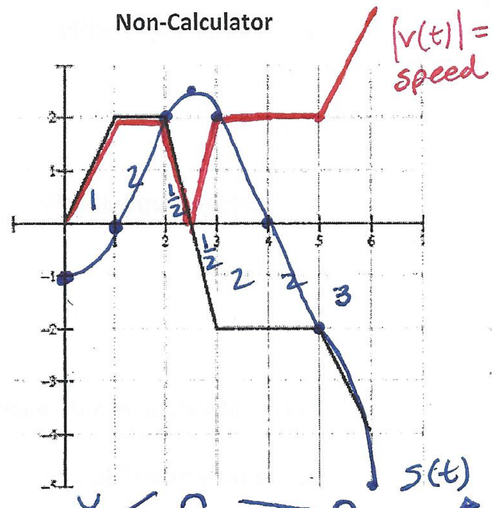
2. Is the speed of the particle increasing or decreasing at time, $t = 2$? Justify your answer.
   $v(2) = -0.436 < 0$ & $a(2) = -0.132 < 0$
   Since both $v(t) < 0$ & $a(t) < 0$, speed is increasing.

3. Find the time, $t \geq 0$ at which the particle reaches its highest point. Justify your answer.
   At time $t = 0.443$ $v(t)$ changes signs from ⊕ to ⊖.
   So $s(0.433)$ is a maximum.

4. Find the position of the particle at time $t = 2$. Is the particle moving toward the origin or away from the origin at $t = 2$? Justify your answer.
   $s(2) = s(0) + \int_0^2 v(t)\,dt$
   $s(2) = -1 - 0.36068$
   $s(2) = -1.360$ or -1.361
   The particle is to the left of the origin at time $t=0$ and $v(2) < 0$ so the particle is moving away from the origin.

FRQ - Motion **Non-Calculator**

Use the graph of velocity $v(t)$ in $(m/\sec)$ for a particle moving along the *x*-axis on $0 \leq t \leq 6$ to complete the following questions and suppose that $x(0) = -1$.

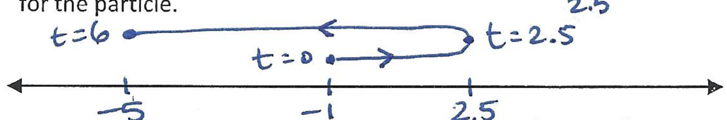


a. Complete the table.
   $X(t) = X(0) + \int_0^t v(t)\,dt$

t	0	1	2	3	4	5	6
$x(t)$	-1	0	2	2	0	-2	-5

2.5

b. Make a horizontal motion graph for the particle.



c. Use a colored pencil to draw the speed graph on $0 \leq t \leq 6$.

d. When is the particle moving left? Moving right? Justify.
   $t \in (2.5, 6)$ b/c $v < 0$ | $t \in (0, 2.5)$ b/c $v > 0$.

e. When is the particle speeding up? Slowing down? Justify
   Speeding up: $v > 0$ & $a > 0$ on $(0,1)$
   $v < 0$ & $a < 0$ on $(2.5, 3)$ $(5,6)$
   Slowing down: $v > 0$ & $a < 0$ on $(2, 2.5)$

f. Write an integral that represents the particles total distance traveled. Evaluate using correct units.
   $\int_0^6 |v(t)|\,dt = 1 + 3.5 + |-7.5| = 11$ meters.

g. Write an equation using an integral that represents the position of the particle at time $t = 6$sec. Evaluate & find $x(6)$. $X(6) = X(0) + \int_0^6 v(t)\,dt = -1 + 3.5 - 7.5 = -5$

h. If the particle the particles position is $s(t)$ representing a height over time, not a position along the x-axis, then draw the position function on the same graph as the velocity. Include accurate concavity behavior.

   $s(t)$ cc up $(0,1)$
   cc down $(2,3)$ $(5,6)$
   linear: increasing $(1,2)$
   linear: decreasing $(3,5)$

MOTION KEY p. 8

AP Calculus AB— Unit 6 Motion with Integrals:
Position – Velocity – Acceleration – Speed – Total Distance – Displacement

FRQ – Integral Defined Functions

Use the figure at the right, sketch the graph of these function

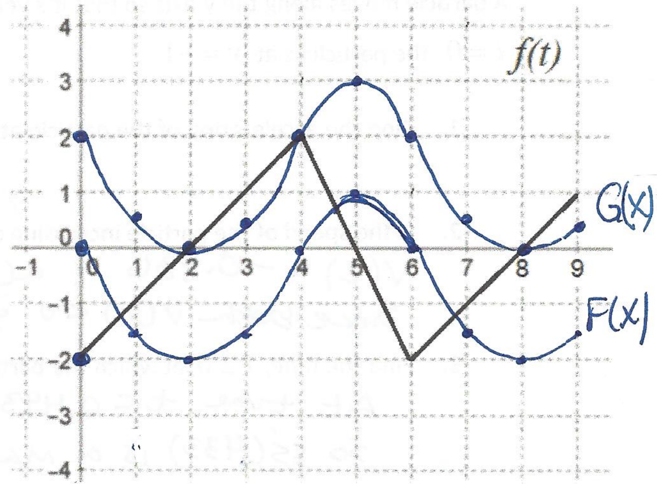
a. $F(x)=\int_0^x f(t)\,dt$

b. $G(x)=\int_2^x f(t)\,dt$

STATE THESE!

$F'=f \quad G'=f$

$F''=f' \quad G''=f'$



Complete the table for each function.

x	0	1	2	3	4	5	6	7	8	9
$F(x)$	0	-1.5	-2	-1.5	0	1	0	-1.5	-2	-1.5
$G(x)$	2	$\frac{1}{2}$	0	$\frac{1}{2}$	2	3	2	$\frac{1}{2}$	0	$\frac{1}{2}$

1. On what intervals are these functions

a. increasing? Justify.

on $(2,5)$ $(8,9)$
b/c $F'=f>0$
$G'=f>0$

b. decreasing? Justify.

on $(0,2)$ $(5,8)$
b/c $F'=f<0$
$G'=f<0$

c. concave up? Justify.

on $(0,4)$ $(6,9)$
b/c $F'=f$ is increasing
$\therefore F''=f'>0$

d. concave down? Justify.

on $(4,6)$
b/c $F'=f$ is decreasing
$\therefore F''=f'<0$

2. For what values of x do each of these functions have

a. a maximum? Justify.

$x=5$ b/c
$F'=f$ changes signs $+$ to $-$
$G'=f$ changes signs $+$ to $-$

b. a minimum? Justify.

$x=2,8$ b/c
$F'=f$ changes signs $-$ to $+$
$G'=f$ changes signs $-$ to $+$

c. point of inflection? Justify

$x=4, 6$ b/c
$F'=f$, $G'=f$
changes from inc to dec at $x=4$
& from dec to inc. at $x=6$.

3. For what values of x do each of these functions have

ABS. occur at either endpts or relatives.

a. an absolute maximum? Justify.

ABS MAX at
$x=5$ b/c
$F(5)=\int_0^5 f(t)\,dt=1$
$G(5)=\int_2^5 f(t)\,dt=3$
are greater than endpoint values $F(0)$ or $F(9)$ & $G(0)$ & $G(9)$ respectively.

b. an absolute minimum? Justify.

ABS MIN at
$x=2$ b/c
$F(2)=\int_0^2 f(t)=-2$
$G(2)=\int_2^2 f(t)\,dt=0$
are less than endpt values.

CHECK "RELATIVES" & "ENDPOINTS"

$F(2)=-2$ $F(5)=1$	$F(0)=\int_0^0 f(t)\,dt=0$ $F(9)=\int_0^9 f(t)\,dt=-1.5$
$G(2)=0$ $G(5)=3$	$G(0)=\int_2^0 f(t)\,dt=2$ $G(9)=\int_2^9 f(t)\,dt=\frac{1}{2}$