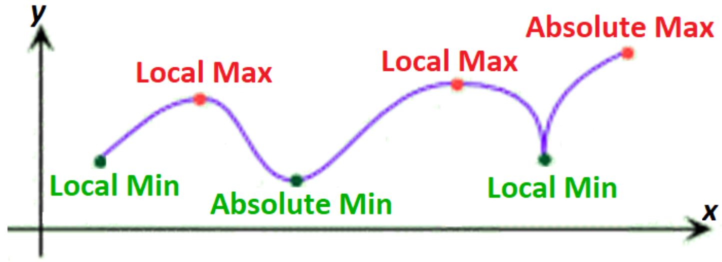
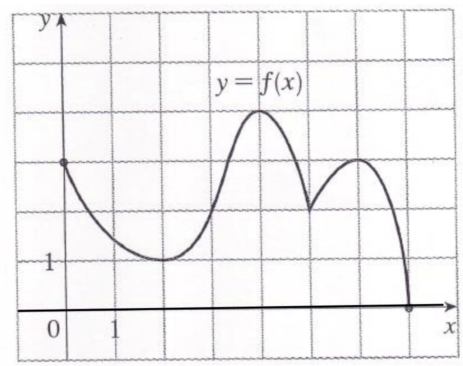
§ 4.1 & § 4.2—Student Notes—Using the First and Second Derivatives

Definition A function *f* has an **absolute maximum** (or **global maximum**) at *c* if $f(c) \geq f(x)$ for all *x* in *D*, where *D* is the domain of *f*. The number $f(c)$ is called the **maximum value** of *f* on *D*. Similarly, the function *f* has an **absolute minimum** (or **global minimum**) at *c* if $f(c) \leq f(x)$ for all *x* in *D* and the number $f(c)$ is called the **minimum value** of *f* on *D*. The maximum and minimum values of *f* are called the **extreme values** of *f*.

Definition A function *f* has an **local maximum** (or **relative maximum**) at *c* if $f(c) \geq f(x)$ when *x* is near *c*. [This means that $f(c) \geq f(x)$ for all *x* in some open interval containing c.] Similarly, the function *f* has an **local minimum** at *c* if $f(c) \leq f(x)$ when *x* is near *c*.



Example 1: Use the graph to state the absolute and local max/min values



Example 2: Describe the maximum and minimum, local and absolute, for the following functions:

a. $f(x) = \cos x$

b. $f(x) = x^2$

c. $f(x) = x^3$

d. $f(x) = |x|$

Definition A critical number of a function *f* is a number *c* in the domain of *f* such that either $f'(c) = 0$ or $f'(c)$ does not exist (DNE).

Theorem If *f* has a local maximum or minimum at *c*, and if $f'(c)$ exists, then $f'(c) = 0$.

Example 3: Find the critical numbers of $f(x) = x^{\frac{3}{5}}(4-x)$.

Increasing/Decreasing Test

(a) If $f'(x) > 0$ on an interval, then *f* is increasing on that interval

(b) If $f'(x) < 0$ on an interval, then *f* is decreasing on that interval

First Derivative Test Suppose that *c* is a critical number of a continuous function *f*.

(a) If f' changes from positive to negative at *c*, then *f* has a local maximum at *c*.

(b) If f' changes from negative to positive at *c*, then *f* has a local minimum at *c*.

(c) If f' does not change sign at *c*, (that is, f' is positive on both sides of *c* or negative on both sides), then *f* has no local maximum or minimum at *c*.

Example 4: Use calculus to find the absolute and relative minimum and maximum values of the function $f(x) = \dfrac{\ln x}{x}, \ on \quad [1,3]$ then check your results using your calculator.

Definition If the graph of *f* lies above all of its tangents on an interval *I*, then it is called **concave upward** on *I*. If the graph of *f* lies below all of its tangents on an interval *I*, then it is called **concave downward** on *I*.

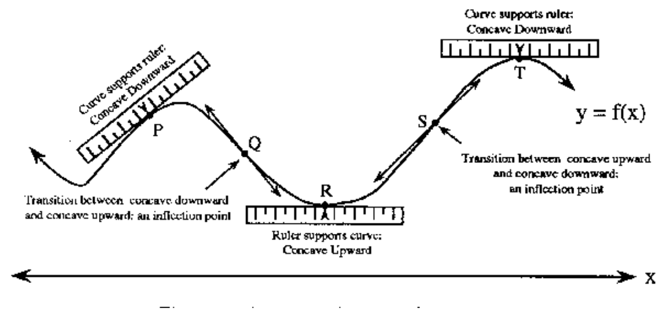
Concavity Test

(a) If $f''(x) > 0$ for all *x* on *I*, then the graph of *f* is concave upward on *I*.

(b) If $f''(x) < 0$ for all *x* on *I*, then the graph of *f* is concave downward on *I*.

Test for Concavity:

A function --- is concave up when f ''(x) > 0
--- is concave down when f ''(x) < 0
--- has no concavity when f ''(x) = 0
--- may have a possible point of inflection if f ''(x) = 0.
--- will have a point of inflection if f ''(x) = 0 and changes signs.

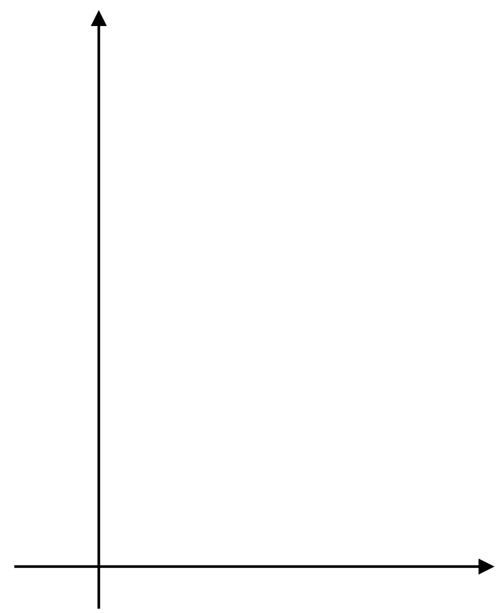


Second Derivative Test Suppose f'' is continuous near *c*.

(a) If $f'(c)=0$ and $f''(c)>0$, then *f* has a local minimum at *c*.

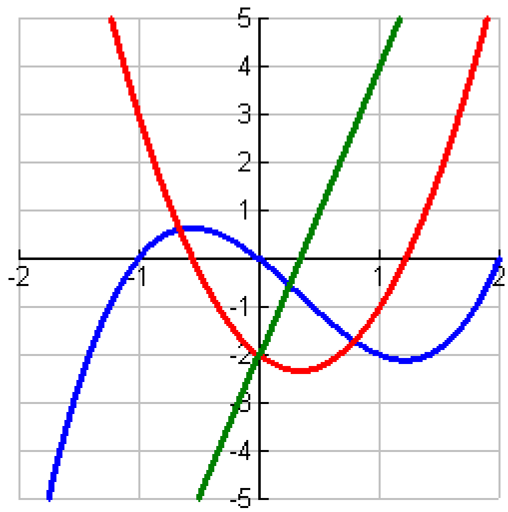
(b) If $f'(c)=0$ and $f''(c)<0$, then *f* has a local maximum at *c*.

Example 5. Given $g(x)=x+2\sin x \quad 0\le x\le 2\pi$, use the second Derivative Test to find the relative extrema and then find the intervals concavity, points of inflection, and use the information to sketch the curve.



Example 6: Given $g(x)=x+2\sin x \quad 0\le x\le 2\pi$ find the intervals concavity, points of inflection, and use the intervals of increasing/decreasing and local maxima and minima to sketch the curve.

§4.1 & §4.2—Day 2—Student Notes—Using the First and Second Derivatives



$f(x) = x^3 - x^2 - 2x$

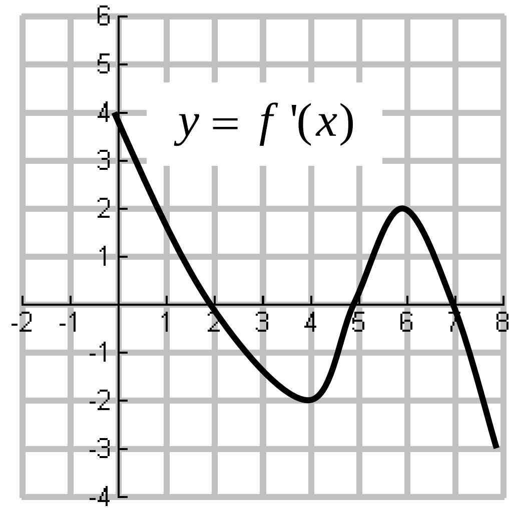
$f'(x) =$ ________________

$f''(x) =$ ________________

f'	f''
1. Positive: f is ____________	1. Positive: f' is ____________ f is ____________
2. Negative: f is ____________	2. Negative: f' is ____________ f is ____________
3. Intercepts: ___________ on f	3. Intercepts: ____________ on f' ____________ on f
4. Max/Min: ____________ on f	
5. Slope is positive: ________ on f _______ on f''	
6. Slope is negative: ________ on f _______ on f''	

Example 1:The graph of the derivative f' of a continuous function f is shown on the interval [0,8]:

a. On what intervals is f increasing or decreasing?



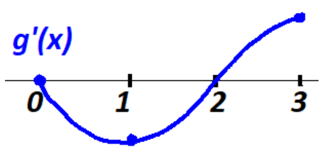
b. At what values of x does f have a local maximum or minimum?

c. On what intervals is f concave upward or downward?

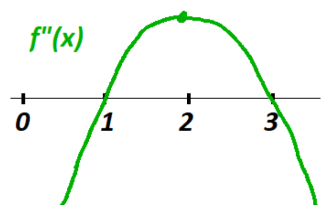
d. State the $x-$coordinates of the points of inflection.

e. Assume that $f(0)=0$, sketch the graph of f.

2. The given graph is g', state several facts about g and g''



3. The given graph is f'', state several facts about f and f'



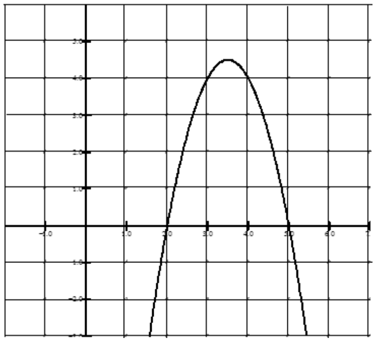
Problems involving the first and second derivative

1) Find the critical numbers of each function:

a) $f(x)=4x^3+2x^2$

b) $f(x)=(x-3)^{\frac{2}{5}}$

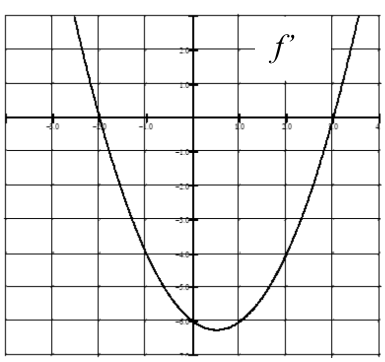
2) Shown is the graph of *f* ’on (1,6). Find the intervals on which f is increasing or decreasing.



3) Find the open intervals on which $f(x) = (x^2 - 9)^{\frac{2}{3}}$ is increasing or decreasing. No calculator.

4) The derivative of a function *f* is given as $f'(x) = \cos(x^2)$. Use a calculator to find the values of x on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ such that f is increasing.

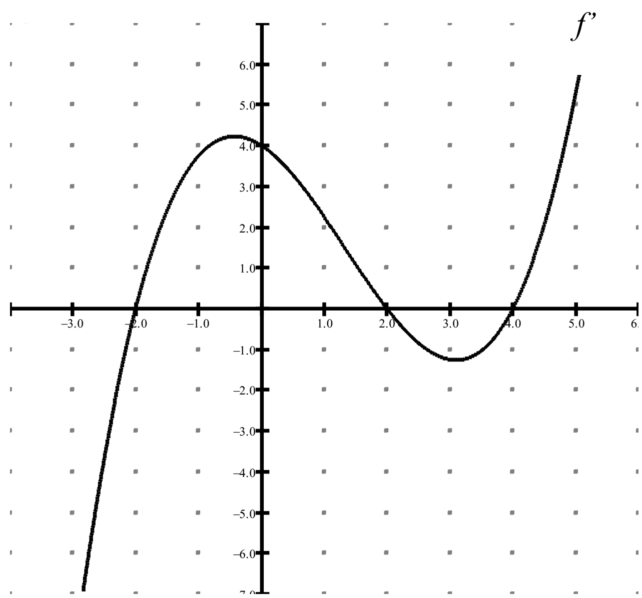
5) The graph of f', the derivative of a function *f* is shown. Find the relative extrema of *f*. Justify your answer.



6) Find the relative extrema of $f(x) = \frac{x^3}{3} - x^2 - 3x$. Use the 2nd Derivative Test. No calculator.

7) Find the relative extrema of $f(x) = (x^2 - 1)^{\frac{2}{3}}$. Use the 1st Derivative Test. No Calculator.

8) The graph of f', the derivative of a function *f* is shown. Find where the function *f* is concave up, where it is concave down and where it has points of inflection.



9) Using a calculator, find the values of *x* at which the graph of $y = x^2 e^x$ changes concavity.

10) Find the points of inflection of the following functions and determine where the function is concave up and where it is concave down. No calculator.

a) $f(x) = x^3 - 6x^2 + 12x - 8$

b) $f(x) = (x-1)^{\frac{2}{3}}$

§4.2 Optimization Practice & HW

Exercises: For each of the following function use derivative techniques to:

I) Find the critical points using derivative techniques and identify them as maxima or minima.
II) Find the absolute maximum and minimum values of each function if it exists.
III) Identify the intervals on which the function is increasing or decreasing.
IV) Find points of inflection and intervals where the functions are concave up or concave down.

SHOW ALL WORK in HW section of your notebook.

1. $f(x) = 4x^3 + 3x^2 - 6x + 1$ on the interval [-2, 1]. Use the first derivative test to justify extrema.

2. $f(x) = \sin x - \cos x$ on the interval [0, π] Use the second derivative test to justify extrema.

3. $f(x) = \sqrt[5]{x^2}$ on the interval [-1, 32] Use the first derivative test to justify extrema.

4. $f(x) = x - \ln x$ on the interval [0.1, 5] Use the second derivative test to justify extrema.

5. $f(x) = x + \dfrac{32}{x^2}$ over all real numbers Use the first derivative test to justify extrema.

6. $f(x) = 2x - e^x$ on the interval [-2, 4] Use the second derivative test to justify extrema.

7. $f(x) = 3x\sqrt[3]{x} - 2x$ on the interval [0, 3] Use the first derivative test to justify extrema.

8. $f(x) = \dfrac{x^4 + 1}{x^2}$ Use the second derivative test to justify extrema.

9. $f(x) = e^x - \ln x^2$ Use the first derivative test to justify extrema.

10. $f(x) = 3\sqrt[3]{x} - 2x$ for the interval [-2, 3] Use the second derivative test to justify extrema.

In exercises 11 and 12, the derivative of the function $y = f(x)$ is given. At what points, if any, does the graph have a relative minimum or relative maximum?

11. $\dfrac{dy}{dx} = (x-1)^2(x-2)$

12. $\dfrac{dy}{dx} = (x-1)^2(x-2)(x-4)$

13. Let $h(x)$ be a function defined for all $x \neq 0$ such that $h(4) = -3$ and the derivative of h is given by $h'(x) = \dfrac{x^2 - 2}{x}$ for all $x \neq 0$.

(a) Find all values of x for which the graph of h has a horizontal tangent, and determine whether h has a local maximum, local minimum or neither at each of these values. Justify your answers.
(b) On what intervals, if any, is the graph of h concave up? Justify your answer.
(c) Write an equation for the tangent line to the graph of h at $x = 4$.
(d) Does the line tangent to the graph of h at $x = 4$ lie above or below the graph of h for $x > 4$? Why?

<u>§4.3 Student Notes -- Optimization & Modeling</u>

<u>Example 1</u> *Finding the maximum volume.* An open box having a square base is to be constructed from 108 in^2 of material. What dimensions will produce a box with maximum volume?

Step 1 State what quantity you are trying to optimize.

Step 2 Make a sketch if one is not provided. Label it.

Step 3 Write a formula for the quantity you want to optimize.

Step 4 The formula must be written in terms of a single variable by substituting from a "secondary equation" & rewrite the formula. (This step might not be needed!)

Step 5 Take the derivative of the equation from step 4.

Step 6 Find the critical numbers.
Set the derivative equal to zero and solve.
Throw out those that fall outside of the domain.
State why you are throwing them out.

Step 7 Establish whether or not the critical number(s) correspond to local extrema and be sure to identify whether you have a max or min. (You DON'T want to minimize something you are trying to maximize or vise-versa!)

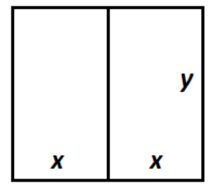
Step 8 Find the value of the quantity to be optimized at each of the relevant critical numbers and answer the original problem in words with units.

§4.3 Optimization Problems: **Solve these on your own paper in your notebook.**

#2:
Find two positive numbers that minimize the sum of twice the first number and the second, if the product of the two numbers is 288.

#3:
What are the dimensions of a closed cylindrical can that can hold 40 cubic inches of liquid and uses the least amount of material?

#4:
A rancher has 200 feet of fencing to enclose two rectangular fields as shown in the diagram. What are the dimensions of the field that maximize the area?



#5:
Find two positive integers whose sum is 20 and whose product is as large as possible.

#6:
An oil can is to be made in the form of a right circular cylinder to contain 1 quart of oil. What dimensions of the can will require the least amount of material? (1 quart liquid = 57.75 cubic inches)

#7:
If you have 100 feet of fencing and you want to enclose a rectangular area adjacent to the long side of the barn, what is the largest area you can enclose?

#8:
The product of two positive numbers is 192. Find the two numbers such that the sum of the first and three times the second is a minimum.

#9:
A right circular cylinder is to be designed to hold 12 fluid ounces using the least amount of material. Find the dimensions of the cylinder. (1 fluid ounce = 1.80469 cubic inches)

#10:
A glass fish tank is to be constructed to hold 72 cubic feet of water. It's base and sides are to be rectangular. The top of the tank is open, of course. The width is 5 ft but the length and depth are variable. Building the tank costs \$10/sqft for the base and \$5/sqft for the lateral sides. Find cost and the dimensions of the tank that minimize the cost.

#11:
Find the point on the parabola $x = -y^2$ that is closest to the point (0,-3).

#12:
Find the area and the dimensions of a rectangle bounded by the function $y = \cos(x)$ and the x-axis whose area is maximized.

#13:
Find the area and dimensions of a rectangle bounded by the curve $y = -x^2 + 4$ and the x-axis and y-axis whose area is maximized.

§4.6 Related Rates—Student Notes

Any equation involving two or more variables that are differentiable functions of time, *t*, can be used to find an equation that relates their corresponding rates.

Identify the formula and then take the derivative with respect to time.

1. $A = \pi r^2$

2. $V = \pi r^2 h$

3. $x^2 + y^2 = r^2$

4. $S = 2\pi rh + 2\pi^2$

5. $A = s^2$

6. $S = 4\pi r^2$

7. $P = 2(l + w)$

8. $V = \frac{1}{3}\pi r^2 h$

9. $V = s^3$

10. $x^2 + y^2 = 25$

11. $C = 2\pi r$

12. $V = \frac{4}{3}\pi r^3$

13. $P = 4s$

14. $A = lw$

15. $A = \frac{1}{2}bh$

Example 1: A hot-air balloon rising straight up from a level field is tracked by a range finder 500 feet from the point of lift-off. At the moment the range finder's elevation angle is $\frac{\pi}{4}$, the angle is increasing at the rate of 0.14 rad/min. How fast is the balloon rising at that moment? (**Answer: 140 ft/min**)

Step 1: Draw a picture and name the variables and constants. Use *t* for time.

Step 2: Write down the numerical information (instantaneous rates of change, constants).

Step 3: Write down what you are trying to find (usually a rate, expressed as a derivative).

Step 4: Write an equation that relates the variables. You may have to combine two or more equations to get a single equation that relates the variable whose rate you want to the variables whose rates you know.

Step 5: Use implicit differentiation with respect to *t* to find the derivatives of the equation in *Step 4.*

Step 6: Now, and only now, substitute your instantaneous rates of change for the variables. The biggest mistake students make is substituting too early.

Step 7: Make sure your answer is reasonable and include units.

§4.6 Related Rates

#1 (**Answer: 140 ft/min**)

Complete these Example Problems and HW problems in your notebook

Example 2: Water runs into a conical tank at the rate of 9 ft^3/min. The tank stands point down and has a height of 10 ft and a base radius of 5 ft. How fast is the water level rising when the water is 6 ft deep?

(**Answer:** $\frac{1}{\pi} \approx 0.318$ **ft/min**)

Example 3: A man 6 ft tall walks at a rate of 5 ft/sec toward a street light that is 16 ft above the ground. At what rate is the length of his shadow changing when he is 10 ft from the base of the light? (**Answer: –3 ft/sec**)

Example 4: A pebble is dropped into a calm pond, causing ripples in the form of concentric circles. The radius, *r*, of the outer ripple is increasing at a constant rate of 1 ft/sec. When this radius is 4 ft, at what rate is the total area of the disturbed water increasing? (**Answer: 8π ft^2/sec**)

Example 5: Gravel is falling in a conical pile at the rate of 100 ft^3/min. Find the rate of change of the height of the pile when the height is 10 ft. Assume that the coarseness of the gravel is such that the radius of the cone is always equal to its height.

HW: Related Rates

HW #1: The width of a rectangle is increasing at a rate of 2 cm/sec and its length is increasing at a rate of 3 cm/sec. At what rate is the area of the rectangle increasing when its width is 4 cm and length is 5 cm? At what rate is the length of the diagonal of the rectangle increasing? At what rate is the perimeter of the rectangle increasing? At what rate is the area of the rectangle increasing?

HW #2: A spherical ball 8 inches in diameter is coated with a layer of ice of uniform thickness. If the ice melts at a rate of 10 in^3/min, how fast is the thickness of the ice decreasing when it is 2 inches thick? How fast is the outer surface area of the ice decreasing at this time?

HW #3: A baseball diamond is a square 90 feet on one side. A runner travels from home plate to first base at 20 ft/sec. How fast is the runner's distance from second base changing when the runner is halfway to first base?

HW #4: A rocket rises vertically from a point on the ground that is 100 m from an observer at ground level. The observer notes that the angle of elevation is increasing at a rate of 12 degrees per second when the angle of elevation is 60 degrees. Find the speed of the rocket at that instant.

HW #5: If $x^2 + y^2 = 25$ and $\frac{dy}{dt} = 6$ cm/sec, find $\frac{dx}{dt}$ when $y = 4$ cm.

HW #6: Sand is being dumped on a pile in such a way that it always forms a cone whose radius equals its height. If the sand is being dumped at a rate of 10 ft^3/min, at what rate is the height of the pile increasing when there is 1000 ft^3 of sand on the pile?

Name____Date________________Period 2 3 4

Applications of Derivatives: **Horizontal Motion**

Class Example: #1) The position $s(t)$ of a particle in motion along a horizontal line at time $t \geq 0$, is given by the equation $s(t) = t^3 - 9t^2 + 15t + 2$. Let $s(t)$ be measured in **meters** and t measured in **seconds**.

a. Find the velocity $v(t)$ of the particle at any time t.

b. Find the acceleration of the particle at any time t.

c. Find all values of t for which the particle is at rest.

d. Find all values of t for which the acceleration zero.

e. State the $\left(t, s\left(t\right)\right)$ coordinates for the t-values in part (c) and (d).

f-g. Sketch a #line for v(t) & a(t)

$v(t) = s'(t)$

$a(t) = s''(t)$

h. If the **<u>velocity of a particle is positive</u>** then the position of the particle is_________________. When we are referring to a particle moving along a horizontal line, if the velocity is positive this means the **particle is moving** _______________.

If the **<u>velocity of a particle is negative</u>** then the position of the particle is_________________. When we are referring to a particle moving along a horizontal line, if the velocity is negative this means the **particle is moving** ________________.

i. State the t-intervals for which the particle is moving to the right & give a reason why?

j. State the t-intervals for which the particle is moving to the left & give a reason why?

k. When the velocity and the acceleration are the **<u>same sign</u>**, either both positive or both negative, then the particle is **<u>speeding up</u>** or accelerating. When the velocity and the acceleration are the **<u>opposite signs</u>**, then the particle is **<u>slowing down</u>** or decelerating.

l. State the t-intervals for which the particle is **<u>speeding up</u>** & give a reason why?

m. State the t-intervals for which the particle is **<u>slowing down</u>** & give a reason why?

n. Draw a horizontal motion diagram

o. Find the **<u>total distance traveled</u>** by the particle on the interval $t \in \left(0, 3\right)$.

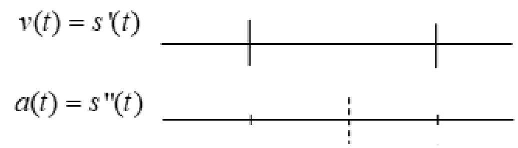
p. Find the **<u>displacement</u>** of the particle on the interval $t \in \left(0, 3\right)$.

#2) The position $s(t)$ of a particle in motion along a horizontal line at time $t \geq 0$, is given by the equation $s(t) = -t^3 + 12t^2 - 36t + 30$. $S(t)$ is measured in **feet** and t is measured in **seconds**.

a. Find the velocity $v(t)$ of the particle at any time t.	b. Find the acceleration of the particle at any time t.
c. Find all values of t for which the particle is instantaneously at rest.	d. Find all values of t for which the acceleration zero.
e. State the $\left(t, s\left(t\right)\right)$ coordinates for the t-values in part (c) and (d).	f-g. Sketch a #line for v(t) & a(t) $v(t) = s'(t)$ $a(t) = s''(t)$
i. State the t-intervals for which the particle is moving forward & give a reason why?	j. State the t-intervals for which the particle is moving backward & give a reason why?
l. State the t-intervals for which the particle is **<u>speeding up</u>** & give a reason why?	m. State the t-intervals for which the particle is **<u>slowing down</u>** & give a reason why?
n. Draw a horizontal motion diagram	
o. Find the **<u>total distance traveled</u>** by the particle on the interval $t \in \left(1, 4\right)$.	p. Find the **<u>displacement</u>** of the particle on the interval $t \in \left(1, 4\right)$.

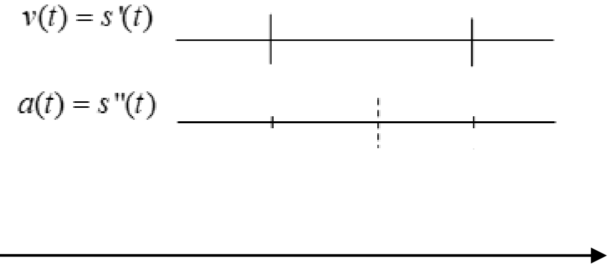
Horizontal Motion Practice Problems:

3. The position of a particle is defined by $x(t)=\frac{8}{3}t^3-11t^2+15t+4$ where $s(t)$ be measured in **meters**, t in **seconds**.

a. Find the velocity $v(t)$ of the particle at any time t.	b. Find the acceleration of the particle at any time t.
c. Find all values of t for which the particle is instantaneously at rest.	d. Find all values of t for which the acceleration zero.
e. State the $(t, s(t))$ coordinates for the t-values in part (c) and (d).	f-g. Sketch a #line for v(t) & a(t) 
i. State the t-intervals for which the particle is moving forward & give a reason why?	j. State the t-intervals for which the particle is moving backward & give a reason why?
l. State the t-intervals for which the particle is **<u>speeding up</u>** & give a reason why?	m. State the t-intervals for which the particle is **<u>slowing down</u>** & give a reason why?
n. Draw a horizontal motion diagram	
o. Find the **<u>total distance traveled</u>** by the particle on the interval $t\in(0,5)$.	p. Find the **<u>displacement</u>** of the particle on the interval $t\in(0,5)$.

4. A particle moves along a horizontal line in such a way that its position at time t is given by $x(t) = t^3 - 12t^2 + 36t - 10$ where x is measured in feet and t in seconds.

a) Find the velocity and acceleration of the particle.
b) Create a first and second derivative number line to help you justify your answers to the questions below.
c) When is the particle moving forward (to the right)?
d) When is the particle moving backward (to the left)?
e) When is the acceleration positive?
f) When is the particle speeding up?
g) When is the particle slowing down?
h) Draw a motion diagram and label it appropriately.



i) Find the **total distance traveled** and the **displacement** of the particle on the interval $t \in (0,5)$
j) Find the maximum velocity of the particle on the interval $t \in (0,5)$.
k) Find the minimum acceleration of the particle on the interval $t \in (0,5)$.

5. A particle is moving on the x-axis. For $t \geq 0$ the particle's position is given by $x(t) = 2t^3 - 13t^2 + 22t - 2$ meters where t is in seconds. Find the intervals when the particle:

a) is moving right,
b) is moving left,
c) has positive acceleration
d) has negative acceleration,
e) speeding up and
f) slowing down,
g) Find the total distance traveled on $t \in (0,4)$
h) Find the displacement on $t \in (0,4)$

6. The position of a particle along a horizontal number line at time t is given by the function $x(t) = -t^2 + 6t - 8$.

a) What is the largest time interval for which x is an increasing function? In which direction is the motion during this time?
b) At what time(s) does the particle change direction? State why you know.
c) On what time interval is the particle slowing down? State why you know.

7. Two particles are moving along a coordinate line. At the end of t seconds their distances from the origin, in feet, are given by $x_1 = 4t - 3t^2$ and $x_2 = t^2 - 2t$, respectively.

a) When do they have the same velocity?
b) If the speed of a particle is the absolute value of its velocity, then when do the two particles have the same speed?
c) When do they have the same position?