Lab 5: Investigating the Intermediate Value Theorem

Goals

- To discover and acquire a feel for one of the major theorems in calculus.
- To apply the theorem in both practical and theoretical ways.
- To understand why continuity is required for the theorem.

In the Lab

This lab will motivate you to discover an important general theorem of calculus. The theorem, called the Intermediate Value Theorem (IVT for short—its name is a clue to what it says), has a clear geometric interpretation but requires a careful mathematical formulation. This lab will develop your intuition for both aspects of the theorem.

1. Consider the function f defined on the closed interval $[1,2]$ by $f(x) = x^5 - 4x^2 + 1$.
   a. Compute the numerical values of $f(1)$ and $f(2)$. Explain why you think the graph of f must cross the x-axis somewhere between $x = 1$ and $x = 2$. That is, why must there be a number c between 1 and 2 such that $f(c) = 0$?
   b. Plot the graph of the function f on $[1,2]$ to support what you said in part a above. Also, estimate any zeros of f (points c at which $f(c) = 0$) on $[1,2]$.
2. Consider the function g defined on the closed interval $[-3,2]$ by

$$g(x) = \begin{cases} x^2 + 1, & \text{for } -3 \le x \le 0, \\ 1 - x^2 - x^4, & \text{for } 0 < x \le 2. \end{cases}$$

   a. Compute the numerical values of $g(-3)$ and $g(2)$. Explain why you think the graph of g must cross the x-axis somewhere between $x = -3$ and $x = 2$.
   b. Plot the graph of g on $[-3,2]$ and estimate any zeros of g.
   c. Now replace g by the slightly altered function h defined on $[-3,2]$ by

$$h(x) = \begin{cases} x^2 + 1, & \text{for } -3 \le x \le 0, \\ -1 - x^2 - x^4, & \text{for } 0 < x \le 2. \end{cases}$$

Once again evaluate the function at its endpoints and think about whether it will take on the value 0 somewhere in the interval. Also plot the graph, and try to explain in general terms why you came to your conclusion.

3. We are now ready to formulate a statement of the Intermediate Value Theorem. Based upon the observations above, fill in the blanks to complete the following.

   Given a ______________ function f defined on the closed interval $[a,b]$ for which 0 is between __________ and __________, there exists a point c between __________ and __________ such that ______________.

Important Note: The above statement of the IVT is an example of what is called an "existence theorem." It says that a certain point exists, but does not give a rule or algorithm for how to find it. (Lovely algorithms for approximating such a point to any desired degree of accuracy do exist.) In the applications of the theorem, the key fact is that such a point exists, not its specific value.

Some Typical Applications

4. *Friendly* warm-up. Use the IVT to prove that the function defined by $f(x) = \sin x^2 - \cos x$ has a zero (that is, a point c at which $f(c) = 0$) on the interval $[0,1]$. Use graphing and/or root finding methods to estimate a zero on the interval.
5. Use the IVT to prove that the function defined by $f(x) = x^3 - e^x + e^{-x}$ has at least five zeros on the real line. Hint: On $[-5,5]$ find six points at which f has alternate signs. Then apply the IVT on five appropriately chosen intervals. Note that a single application of IVT on $[-5,5]$ guarantees only one zero, but it does not rule out the possibility of more than one zero. Use graphing and/or root finding methods to estimate the five zeros of f.
6. If your oven is at 250° and you turn it off, is there ever an instant when the oven temperature is 170°? Explain your answer and its relation to the IVT.
7. If you remove marbles from a bag one at a time, must there always come a time when the bag contains exactly half the number of marbles it began with? Again, explain your answer and its relation to the IVT.

8. In your notebook sketch the graph of a continuous function g over the interval $[0,1]$. Now draw the graph of another continuous function h over the same interval with the property that $h(0) < g(0)$ and $h(1) > g(1)$.

   a. Must these two graphs cross? Express this behavior in terms of a condition involving the functions g and h and a point c in the interval $(0,1)$.

   b. Give a proof that what you observed in part a must always be true for any two continuous functions g and h on $[0,1]$ with the property that $g(0) > h(0)$ and $h(1) > g(1)$. Hint: Apply the IVT thoughtfully to the function $f = g - h$.

   c. One plate has been in the freezer for a while, the other is in a warm oven. The locations of the two plates are then switched. Will there be a moment when the plates are at the same temperature? How does your answer relate to the ideas developed in parts a and b above?

   d. Show that on any circle (such as a great circle on the surface of the earth) there is always a pair of points at the opposite ends of a diameter that are at the same temperature.

9. *Fixed point theorem.* A *fixed point* of a function f is a point c in the domain of f for which $f(c) = c$.

   a. In your notebook, sketch the graph of a continuous function f defined on an interval $[a,b]$ and whose values also lie in the interval $[a,b]$. You may choose the endpoints a and b of the interval arbitrarily, but be sure that the image of the function is contained within its domain. Locate a point c on your graph that is a fixed point of f.

   b. Try to draw the graph of a continuous function f defined on an interval $[a,b]$ with values in the interval $[a,b]$ that has no fixed points. What is getting in your way? Prove that any continuous function f defined on $[a,b]$ with values in $[a,b]$ must have a fixed point c in the interval $[a,b]$. Hint: Use your pictures for guidance and draw in the graph of the line $y = x$. Ask your instructor for further hints if you are still stuck.

Further Exploration

10. *A more general IVT.* There is nothing special, as far as the IVT is concerned, about a function having the value 0.

    a. Give an intuitive reason why the function $g(x) = x^5 - 4x^2 + 1$ must take on the value 7 somewhere in the interval $[1,2]$. Hint: Compute $g(1)$ and $g(2)$. Prove this fact by applying the IVT to the function $f(x) = g(x) - 7$ on the interval $[1,2]$. What other values must the function g achieve on $[1,2]$?

    b. Generalize the idea behind part a to complete, with understanding, the following statement.

    General Intermediate Value Theorem: Suppose the function g is __________ on the closed interval $[a,b]$. For any real number d between ________ and ________ there exists a point c between ________ and ________ such that __________.

11. Give a specific example of a function f that is not continuous on a closed interval $[a,b]$, but for which the conclusion of the General IVT on that interval $[a,b]$ still holds. Why does this not contradict the theorem?