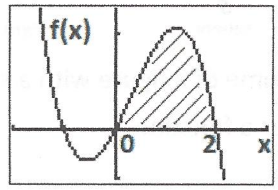
Unit 6: Chapter 8 – Areas and Volumes

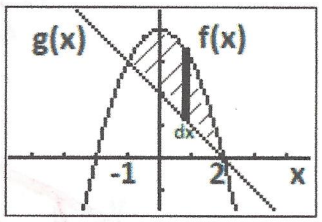
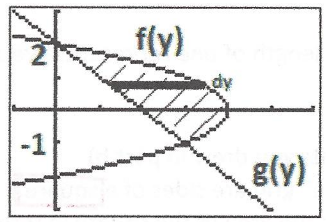
Part 1: Areas To find the area bounded by a function $f(x)$ and the $x-axis$ on an interval from $x=a$ and $x=b$ we use the integral: $\int_a^b f(x)\,dx$. This is consistent with $\int_a^b (upper - lower)\,dx$ since

$$\int_a^b f(x)\,dx = \int_a^b [f(x)-0]\,dx.$$



For example: → $\int_0^2 (2x+x^2-x^3)\,dx = \int_0^2 (-x)(x+1)(x-2)\,dx$

The examples below show the two options for the integral set up to find areas between curves.

On the x-interval from $x=a$ and $x=b$:	On the y-interval from $y=c$ and $y=d$:
$\int_a^b (upper\ curve - lower\ curve)\,dx$  $\int_{-1}^{2} (f(x)-g(x))\,dx = \int_{-1}^{2} ((4-x^2)-(2-x))\,dx$	$\int_c^d (right\ curve - left\ curve)\,dy$  $\int_{-1}^{2} (f(y)-g(y))\,dy = \int_{-1}^{2} ((4-y^2)-(2-y))\,dy$

When the area is bounded by two curves, you may need to solve for the intersection points to find the limits of integration. For efficiency and accuracy, it is best to store the intersection values on the calculator. STO>A immediately then repeat and STO>B immediately.

DAY 115 HW: (Fix.) Draw and label an accurate graph, then either write an integral to find the area between the two functions from $x=a$ and $x=b$ <u>or</u> to find the area bounded by the two functions. For bounded regions, you must determine the limits of integration. Show work to evaluate the integral and find the area. Do your best to graph the functions without your graphing calculator when indicated "no calculator".

1. between $f(x)=x+1$ $g(x)=\frac{1}{2}x$ $a=0,\ b=3$ No calculator $\frac{21}{4}$	2. between $f(x)=4-x$ $g(x)=x^2-x-2$ $a=-1,\ b=2$ No calculator 15	3. bounded by $f(x)=4-x^2$ $g(x)=\frac{1}{4}x^3-x$ No calculator $\frac{32}{3}$	4. bounded by $f(x)=e^x$ $g(x)=2-x^2$ Use calculator to find limits of integration. 1.452
5. bounded by $f(x)=\cos(x)$ $g(x)=\sin(x)$ $2\sqrt{2}$ Limits of integration may vary. No calculator	6. bounded by $f(x)=\sqrt{x}$ $g(x)=x^2$ $\frac{1}{3}$ Integrate with respect to x. No calculator	7. bounded by $f(x)=\sqrt{x}$ $g(x)=x^2$ $\frac{1}{3}$ Integrate with respect to y. No calculator	8. bounded by $f(x)=\cos(x)$ $g(x)=\sin(x)$ & $y-axis$ $\sqrt{2}-1$ No calculator
9. bounded by $y=x-1$ & $x=(y-1)^2$ Integrate with respect to y. $\frac{9}{2}$		10. bounded by $f(x)=\sqrt{x}$ & $g(x)=2\sqrt{5-x}$ Integrate with respect to y. $\frac{20}{3}$	

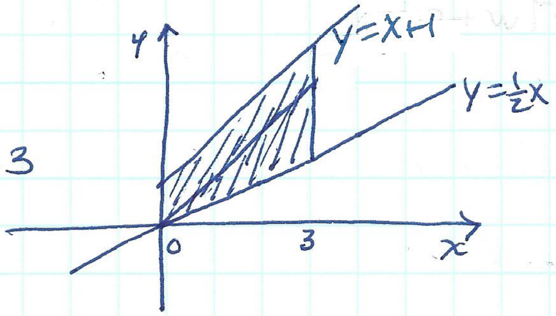
DAY 115 CH 8 AREAS between curves

P. 1 #1-10

① $f(x) = x+1$

$g(x) = \frac{1}{2}x$

$a=0 \quad b=3$



$\int_0^3 (x+1) - (\frac{1}{2}x)\,dx$

$\int_0^3 (\frac{1}{2}x + 1)\,dx$

$\frac{1}{4}x^2 + x \Big|_0^3$

$(\frac{9}{4} + 3) - (0) = \boxed{\frac{21}{4}}$

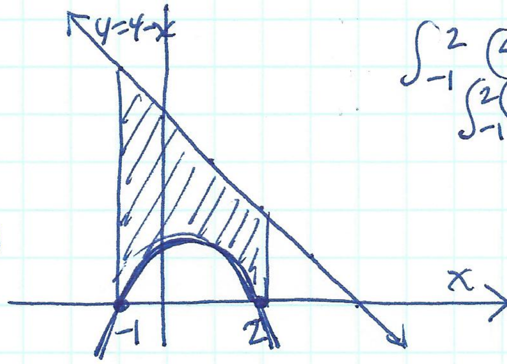
② $f(x) = 4-x$

$g(x) = x^2 - x - 2$

$a=-1, b=2$

$g(x) = (x-2)(x+1)$

zeros: $x = 2, 1$



$\int_{-1}^2 (4-x) - (x^2 - x - 2)\,dx$

$\int_{-1}^2 (-x^2 + 6)\,dx$

$-\frac{1}{3}x^3 + 6x \Big|_{-1}^2$

$(-\frac{8}{3} + 12) - (+\frac{1}{3} - 6)$

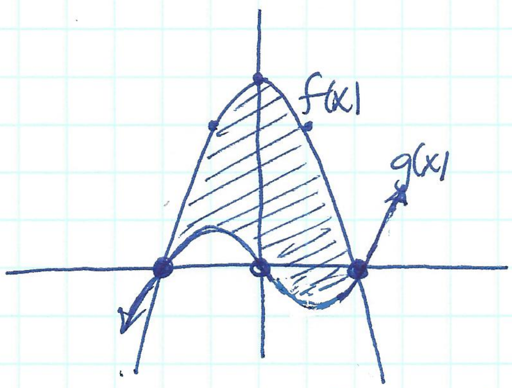
$18 - \frac{9}{3} = \boxed{15}$

③ $f(x) = 4 - x^2$

$g(x) = \frac{1}{4}x^3 - x$

$g(x) = \frac{1}{4}x(x^2 - 4)$

$g(x) = \frac{1}{4}x(x-2)(x+2)$



$\int_{-2}^2 (4-x^2) - (\frac{1}{4}x^3 - x)\,dx$

$\int_{-2}^2 (-\frac{1}{4}x^3 - x^2 + x + 4)\,dx$

$= -\frac{1}{16}x^4 - \frac{1}{3}x^3 + \frac{1}{2}x^2 + 4x \Big|_{-2}^2$

$= (\cancel{-1} - \frac{8}{3} \cancel{+2} + 8) - (\cancel{-1} + \frac{8}{3} \cancel{+2} - 8)$

$16 - \frac{16}{3} = \frac{48-16}{3} = \boxed{\frac{32}{3}}$

b/c $g(x)$ is ODD

SAME as $\int_{-2}^2 (4-x^2)\,dx = 2\int_0^2 (4-x^2)\,dx$

$4x - \frac{1}{3}x^3 \Big|_{-2}^2 \quad = 2(4x - \frac{1}{3}x^2)\Big|_0^2$

$(8 - \frac{8}{3}) - (-8 - \frac{8}{3})$

$2(8 - \frac{8}{3}) = \boxed{\frac{32}{3}}$

④ $f(x) = e^x$

$g(x) = 2 - x^2$

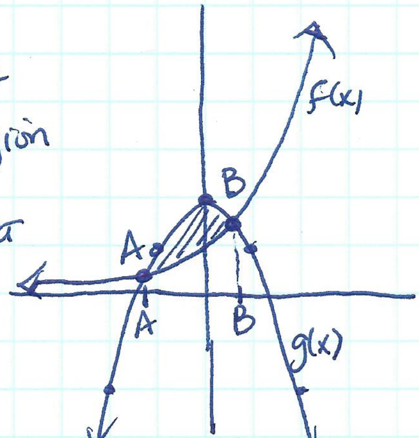
bounded region

2ND CALC INTERSECT

$B = 0.537274...$

$A = -1.315974...$

STORE THESE VALUES!



$\int_A^B (2 - x^2 - e^x)\,dx$

$= 2x - \frac{1}{3}x^3 - e^x \Big|_A^B$

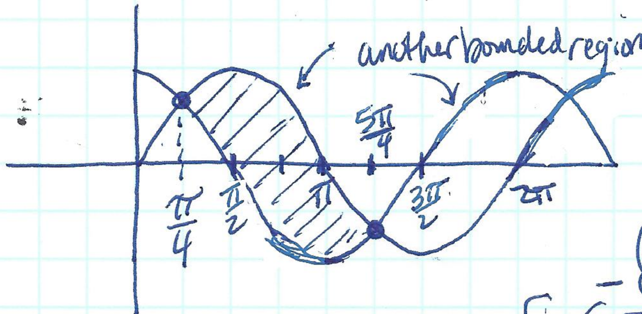
$= (2B - \frac{1}{3}B^3 - e^B) - (2A - \frac{1}{3}A^3 - e^A)$

evaluate on calculator

$= \boxed{1.452}$

DAY 115 CH 8 Areas between curves. P! #1–10

⑤ $f(x) = \cos x$
$g(x) = \sin x$



$$\int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} (\sin x - \cos x)\,dx$$

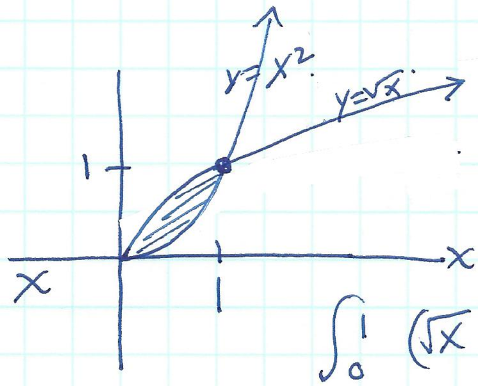
$$\left(-\cos x - \sin x\right)\Big|_{\frac{\pi}{4}}^{\frac{5\pi}{4}}$$
$-(\cos x + \sin x)$

$$\left[-\left(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\right)\right] - \left[-\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right)\right]$$

$+\sqrt{2} + \sqrt{2}$

$= \boxed{2\sqrt{2}} \approx 2.828$

⑥ $f(x) = \sqrt{x}$
$g(x) = x^2$



integrate w/r/t x

$$\int_0^1 (\sqrt{x} - x^2)\,dx = \frac{2}{3}x^{\frac{3}{2}} - \frac{1}{3}x^3 \Big|_0^1$$
$$= \frac{2}{3}(1) - \frac{1}{3}(1) = \boxed{\frac{1}{3}}$$

⑦ Same as #6

integrate w/r/t y.

$$\int_0^1 (\text{right} - \text{left})\,dy$$

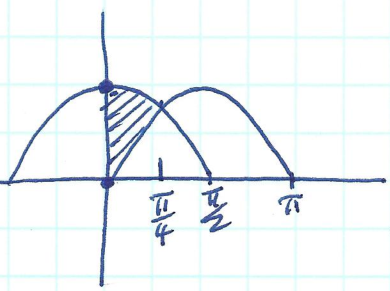
$$\int_0^1 (\sqrt{y} - y^2)\,dy$$ same just in terms of y. ☺

$= \boxed{\frac{1}{3}}$

Right $y = x^2$
$x = \sqrt{y}$

Left $y = \sqrt{x}$
$x = y^2$

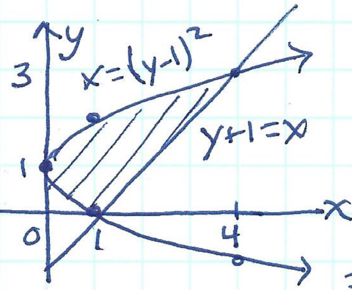
⑧ $f(x) = \cos x$
$g(x) = \sin x$
& y-axis



$$\int_0^{\pi/4} (\cos x - \sin x)\,dx$$
$$= \sin x + \cos x \Big|_0^{\pi/4}$$
$$\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right) - (0 + 1)$$
$\boxed{\sqrt{2} - 1} \approx +0.4142$

⑨ $y = x - 1$
$x = (y-1)^2$

integrate w/r/t y



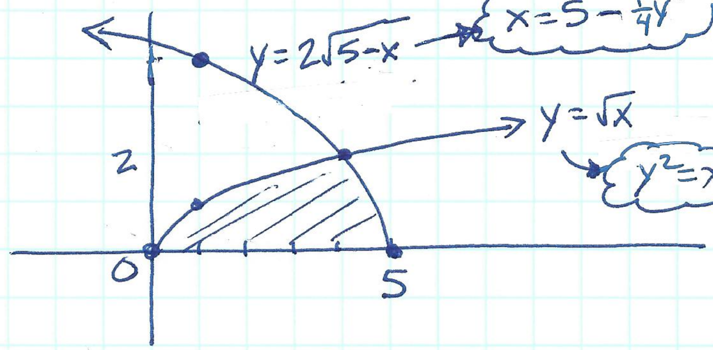
$$\int_0^3 (y+1) - (y-1)^2\,dy$$
$$\left(\frac{1}{2}y^2 + y\right) - \frac{1}{3}(y-1)^3 \Big|_0^3 = F(3) - F(0)$$
$$= \left[\frac{9}{2} + 3 - \frac{8}{3}\right] - \left[0 + \frac{1}{3}\right]$$
$$= \frac{9}{2} + \cancel{3} - \cancel{3} = \boxed{\frac{9}{2}}$$

DAY 115 CH8 Areas between Curves. P1 #1-10

(10) $f(x) = \sqrt{x}$

$g(x) = 2\sqrt{5-x}$

bounded by f & g & x-axis.



* Integrate w/r/t y & write only one integral

$$\int_0^2 \left(5 - \tfrac{1}{4}y^2 - y^2\right) dy = \int_0^2 \left(5 - \tfrac{5}{4}y^2\right) dy$$

$$= 5y - \tfrac{5}{12}y^3 \Big|_0^2$$

$$= \left(10 - \tfrac{10}{3}\right) - (0)$$

$$= \boxed{\tfrac{20}{3}}$$

* Two integrals required if w/r/t x.

$$\int_0^4 \sqrt{x}\,dx + \int_4^5 2\sqrt{5-x}\,dx$$

$$\int_0^4 \sqrt{x}\,dx - \int_4^5 2\sqrt{5-x}\,dx$$ ← U-sub.

$$\tfrac{2}{3}x^{3/2}\Big|_0^4 - 2\cdot\tfrac{2}{3}(5-x)^{3/2}\Big|_4^5$$

$$\left(\tfrac{16}{3} - 0\right) - \tfrac{4}{3}(0 - 1)$$

$$\tfrac{16}{3} + \tfrac{4}{3} = \boxed{\tfrac{20}{3}}$$ ✓