DAY 36

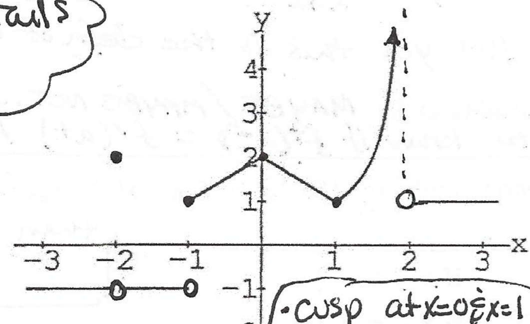
ANSWER KEY

§2.6

WS: **Continuity & Differentiability**

1. The graph of a function f is given below.

Pay attention to details. Be sure for all questions you know what it means to justify or explain.



• cusp at $x=0$ & $x=1$
• sharp corners
• sharp points

a. For what numbers x in $[-3,3]$ is f *not* continuous at x? Explain your answer.

b. For what numbers x in $[-3,3]$ is f *not* differentiable at x? Explain your answer.

1a) f is discontinuous at
- $x=-2$ (nonremovable jump discontinuity) b/c $f(-2)=2 \neq \lim_{x\to -2} f(x)=-1$
- $x=-1$ (nonremovable jump discontinuity) b/c $f(-1)=1 \neq \lim_{x\to -1} f(x)=$ dne
- $x=2$ (infinite discontinuity) b/c $f(2)$ dne $\neq \lim_{x\to 2} f(x)$ dne

1b) $f(x)$ is not differentiable when $f(x)$ is not continuous $\therefore$ @ $x=-2,-1$ & 2. Also at $x=0$ b/c $f'(0^-)\neq f'(0^+)$, $+1\neq -1$, and at $x=1$ b/c $f'(1^-)\neq f'(1^+)$

2. For some given non-zero number a, define

$$f(x)=\begin{cases}(x^2-a^2)/(x-a) & \text{if } x\neq a,\\ 0 & \text{if } x=a.\end{cases}$$

$$f(x)=\begin{cases}x+a, & x\neq a\\ 0, & x=a\end{cases}$$

a. Is f defined at a? Yes $f(a)=0$

b. Does $\lim_{x\to a} f(x)$ exist? Justify your answer. Yes b/c $\lim_{x\to a^-} f(x)=\lim_{x\to a^+} f(x)=2a \therefore \lim_{x\to a} f(x)=2a$ ✓

c. Is f continuous at a? Justify your answer. No b/c $f(a)=0\neq \lim_{x\to a} f(x)=2a$ (hole @ $(a,2a)$ removable discontinuity)

d. Is f differentiable at a? Justify your answer. No b/c $f(x)$ is discontinuous @ $x=a$. Continuity is a prerequisite of differentiability!

3. For $x\neq 0$ define $f(x)=\dfrac{\sin x}{x}$. Is it possible to define $f(0)$ so that f is continuous at 0? Explain your answer.

Yes since $\lim_{x\to 0}\frac{\sin(x)}{x}=1$ but $f(0)$ dne on $f(x)$, redefine $f(x)=\begin{cases}\frac{\sin(x)}{x}, & x\neq 0\\ 1, & x=0\end{cases}$ and now the function is continuous because the hole has been removed... plugged with the point $(0,1)$.

4. If $\lim_{x\to a} f(x)=L$, which of the following statements, if any, MUST be true? answer.

(i) f is defined at a. → not necessarily → $f(a)$ may not exist or may be different than the limit L.

(ii) $f(a)=L$. → not necessarily →

(iii) f is continuous at a. → not if $f(a)$ dne or $f(a)\neq L$.

(iv) f is differentiable at a. → not necessarily... only true if $f(x)$ is continuous at $x=a$ $\therefore$ only true if $\lim_{x\to a} f(x)=\underbrace{L=f(a)}_{\text{\& we don't know this.}}$

* none of these MUST be true

→ this means $\boxed{f(a) = \lim_{x\to a} f(x) = L}$

5. If a function f is continuous at a, which of the following statements, if any, MUST be true? Justify your answer.

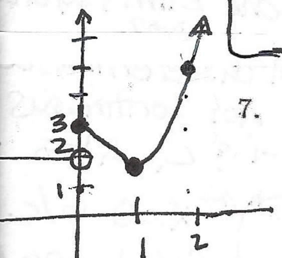
   (i) f is defined at a. yes $f(a) = L$ TRUE

   (ii) $\lim_{x\to a} f(x)$ exists. yes $\lim_{x\to a} f(x) = L$ TRUE

   (iii) $\lim_{x\to a} f(x) = f(a)$ yes this is the defn of continuity TRUE

   (iv) f is differentiable at a. MAYBE/ MAYBE NOT... I don't know for sure b/c I need to know if $f'(a^-) = f'(a^+)$ to determine differentiability.

6. Suppose f is continuous and $\lim_{x\to a} f(x) = L$. Find $f(a)$. Justify your answer. If $f(x)$ is continuous then by defn of continuity $\lim_{x\to a} f(x) = L = f(a)$ $\therefore$ $f(a) = L$



7. Let $f(x) = \begin{cases} 2 & \text{if } x < 0, \\ 3 - x & \text{if } 0 \le x \le 1, \\ x^2 + 1 & \text{if } x > 1. \end{cases}$

   a. Is f continuous at $x = 0$? Justify your answer. NO → f is NOT continuous at $x = 0$ b/c $f(0) = 3$ & $\lim_{x\to 0} f(x)$ dne.

   b. Is f continuous at $x = 1$? Justify your answer. Yes → f is continuous at $x = 1$ b/c $f(1) = 2$ & $\lim_{x\to 1} f(x) = 2 = f(1)$

8. Determine whether the following statements MUST be true or are at least SOMETIMES false. Justify your answers.

   a. If f is continuous at a point x ($x \to c$), then it is differentiable at x. ONLY SOMETIMES: look @ $y = |x|$ If $f(x)$ has a sharp point @ $x = 0$ so that $f'(0^-) \ne f'(0^+)$ then f is not differentiable. If $f(x)$ is smooth like a polynomial so that $f'(c^-) = f'(c^+)$ then $f(x)$ is differentiable.

   b. If f is differentiable at a point x, then it is continuous at x. ALWAYS f can only be differentiable if $f(x)$ is first known to be continuous!

9. Let $f(x) = \begin{cases} ax & \text{if } x \le 1, \\ bx^2 + x + 1 & \text{if } x > 1. \end{cases}$

   See separate paper.

   a. Find all choices of a and b such that f is continuous at $x = 1$.

   b. Draw the graph of f when $a = 1$ and $b = -1$.

   c. Find values of a and b such that f is differentiable at $x = 1$.

   d. Draw the graph of f for the values of a and b found in part c.

10. George takes a trip from St. Louis to Chicago. He leaves at 9 AM on Monday and arrives at 2 PM that day. He returns on Tuesday, leaving at 9 AM and arriving back in St. Louis at 2 PM, retracing exactly the same route. Show that there is a point on the road through which he passes at the same time both days.

11. Determine whether the following statements are always true or are at least sometimes false. Justify your answers.

    See paper

    a. If $f(1) < 0$ and $f(2) > 0$, then there must be a point z in $(1, 2)$ such that $f(z) = 0$.

    b. If f is continuous on $[1, 2]$, $f(1) < 0$ and $f(2) > 0$, then there must be a point z in $(1, 2)$ such that $f(z) = 0$.

    c. If f is continuous on $[1, 2]$ and there is a point z in $(1, 2)$ such that $f(z) = 0$, then $f(1)$ and $f(2)$ must have different signs.

    d. If f has no zeros and is continuous on $[1, 2]$, then $f(1)$ and $f(2)$ have the same sign.

DAY 36

W.S. Continuity & Differentiability (continued) — ANSWERS

(9) $f(x) = \begin{cases} ax & , x \le 1 \\ bx^2+x+1 & , x > 1 \end{cases}$

a) If $f(x)$ is continuous then $f(1) = \lim_{x\to1} f(x)$

- $f(1) = 1a$
- $\lim_{x\to1^-} (ax) = 1a$ & $\lim_{x\to1^+} (bx^2+x+1) = b+2$

$\therefore \lim_{x\to1} f(x)$ means that LHL = RHL to exist.

$a = b+2$

So all a & b values that satisfy $\boxed{a=b+2}$ will make $f(x)$ continuous.

b) If $a=1$ & $b=-1$

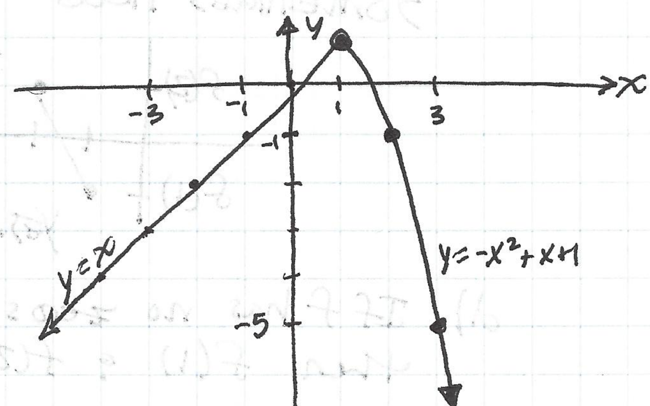
$f(x) = \begin{cases} x & , x \le 1 \\ -x^2+x+1 & , x > 1 \end{cases}$

$-(x^2-x+\frac14)+1+\frac14$

$-(x-\frac12)^2+\frac54$

$V(\frac12, \frac54)$

opening down.



c) For $f(x)$ to be differentiable LHslope = RHslope on $f(x)$

For $f(x)$ to be continuous $a=b+2$

Solve system

$a=b+2$

$a=2b+1$

$b+2=2b+1$

$1=b$

$\therefore a=3$

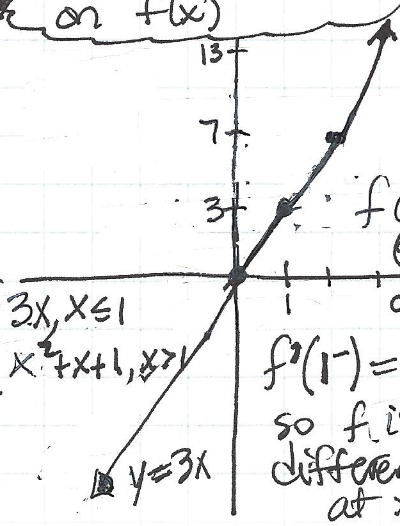
$f'(x) = \begin{cases} a & , x \le 1 \\ 2bx+1 & , x > 1 \end{cases}$

$a = 2bx+1 \,\Big|_{x=1}$

$a = 2b+1$

So any value for a & b such that $a=2b+1$ will make $f(x)$ differentiable

$f(x) = \begin{cases} 3x, & x \le 1 \\ x^2+x+1, & x>1 \end{cases}$



f continuous @ $x=1$ and $f'(1^-) = f'(1^+)$ so f is also differentiable at $x=1$

(11) ALWAYS TRUE SOMETIMES TRUE NEVER TRUE.

a) If $f(1) < 0$ & $f(2) > 0$ then there must be a point $x = z$ on the interval $(1, 2)$ such that $f(z) = 0$

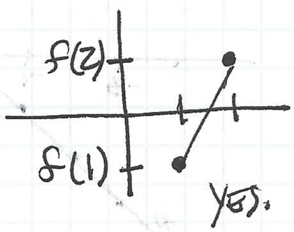
SOMETIMES TRUE if $f(x)$ is CONTINUOUS then the Intermediate value theorem guarantees a zero when $f(x)$ changes sign on the interval.

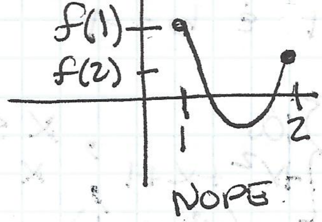
b) If f is continuous on $[1,2]$, $f(1) < 0$ & $f(2) > 0$, then $\exists$ x on $(1,2)$ such that $f(x) = 0$

ALWAYS TRUE & guaranteed by the I.V.T.

c) If f is continuous on $[1,2]$ & there is a point x on $(1,2)$ such that $f(x) = 0$ then $f(1)$ & $f(2)$ have different signs.

SOMETIMES TRUE example graphs.





d) If f has no zeros and is continuous on $[1,2]$ then $f(1)$ & $f(2)$ have the same sign

TRUE b/c f is continuous & has no zeros on $[1,2]$ the graph of $f(x)$ must either be above the x-axis on $[1,2]$ or below the x-axis on $[1,2]$.