

Day 71

§4.6 Related Rates—Student Notes

Any equation involving two or more variables that are differentiable functions of time, t , can be used to find an equation that relates their corresponding rates.

Related Rates

Identify the formula and then take the derivative with respect to time.

1. $A = \pi r^2$
 CIRCLE
 area $\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$

2. $V = \pi r^2 h$
 cylinder
 volume $\frac{dV}{dt} = \pi \left(2rh \frac{dr}{dt} + r^2 \frac{dh}{dt} \right)$

3. $a^2 + b^2 = c^2$
 Pythagorean
 Thm $a \frac{da}{dt} + b \frac{db}{dt} = c \frac{dc}{dt}$

4. $S = 2\pi rh + 2\pi r^2$
 Surface area
 cylinder $\frac{dS}{dt} = 2\pi \left(\frac{dr}{dt} h + h \frac{dr}{dt} \right) + 4\pi r \frac{dr}{dt}$

5. $A = s^2$
 Area of
 square $\frac{dA}{dt} = 2s \frac{ds}{dt}$

12. $V = \frac{4}{3} \pi r^3$
 volume sphere $\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$

6. $S = 4\pi r^2$
 Surface area
 sphere $\frac{dS}{dt} = 8\pi r \frac{dr}{dt}$

13. $P = 4s$
 perimeter square $\frac{dP}{dt} = 4 \frac{ds}{dt}$

7. $P = 2l + 2w$
 perimeter
 rectangle $\frac{dP}{dt} = 2 \frac{dl}{dt} + 2 \frac{dw}{dt}$

14. $A = lw$
 Area of rectangle
 P-gram. $\frac{dA}{dt} = \left(\frac{dl}{dt} \cdot h + l \cdot \frac{dh}{dt} \right)$

8. $V = \frac{1}{3} \pi r^2 h$
 volume cone $\frac{dV}{dt} = \frac{\pi}{3} \left(2rh \frac{dr}{dt} + r^2 \frac{dh}{dt} \right)$

15. $A = \frac{1}{2} bh$
 Area of triangle $\frac{dA}{dt} = \frac{1}{2} \left(\frac{db}{dt} h + b \frac{dh}{dt} \right)$

9. $V = s^3$
 volume cube $\frac{dV}{dt} = 3s^2 \frac{ds}{dt}$

10. $x^2 + y^2 = 25$
 circle w/ radius 5 $2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0$
 $x \frac{dx}{dt} + y \frac{dy}{dt} = 0$

11. $C = 2\pi r$
 circumference
 circle. $\frac{dC}{dt} = 2\pi \frac{dr}{dt}$

#1

DAY 71

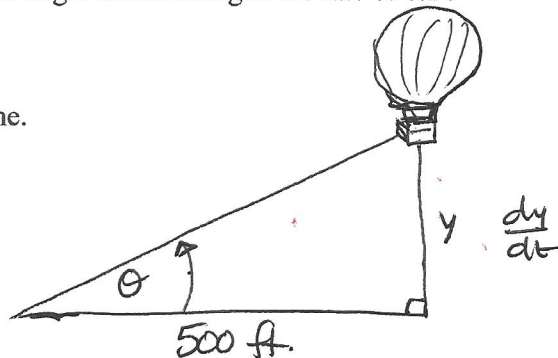
Example 1: A hot-air balloon rising straight up from a level field is tracked by a range finder 500 feet from the point of lift-off. At the moment the range finder's elevation angle is $\frac{\pi}{4}$, the angle is increasing at the rate of 0.14 rad/min. How fast is the balloon rising at that moment?

Step 1: Draw a picture and name the variables and constants. Use t for time.

$$\theta = \frac{\pi}{4} \quad \frac{d\theta}{dt} = 0.14 \frac{\text{rad}}{\text{min}} \quad \text{find } \frac{dy}{dt}$$

↓

$$\therefore y = 500 \text{ ft}$$



Step 2: Write down the numerical information (instantaneous rates of change, constants).

$$\theta = \frac{\pi}{4} \quad \frac{d\theta}{dt} = 0.14 \frac{\text{rad}}{\text{min}}$$

$$\therefore y = 500 \text{ ft.}$$

Step 3: Write down what you are trying to find (usually a rate, expressed as a derivative).

$$\frac{dy}{dt} = ?$$

should be positive since balloon is rising.

Step 4: Write an equation that relates the variables. You may have to combine two or more equations to get a single equation that relates the variable whose rate you want to the variables whose rates you know.

$$\tan \theta = \frac{y}{500}$$

$$\text{or } 500 \tan \theta = y$$

Step 5: Use implicit differentiation with respect to t to find the derivatives of the equation in Step 4.

$$500 \sec^2 \theta \frac{d\theta}{dt} = \frac{dy}{dt} \quad \text{or } \sec^2 \theta \frac{d\theta}{dt} = \left(\frac{1}{500}\right) \left(\frac{dy}{dt}\right)$$

↙ solve for $\frac{dy}{dt}$

Step 6: Now, and only now, substitute your instantaneous rates of change for the variables. The biggest mistake students make is substituting too early

$$500 \sec^2\left(\frac{\pi}{4}\right) (0.14) = \frac{dy}{dt}$$

$$\frac{dy}{dt} = 500 (\sqrt{2})^2 (0.14) \quad \therefore \frac{dy}{dt} = 140 \frac{\text{ft.}}{\text{min.}}$$

Step 7: Make sure your answer is reasonable and include units.

The balloon is rising at a rate of 140 ft/min when the angle of elevation from the range finder is $\theta = \frac{\pi}{4}$.

DAY

§4.6 Related Rates

Complete these Example Problems and HW problems in your notebook

71

Example 2: Water runs into a conical tank at the rate of $9 \text{ ft}^3/\text{min}$. The tank stands point down and has a height of 10 ft and a base radius of 5 ft. How fast is the water level rising when the water is 6 ft deep?

(Answer: ft/min)

Example 3: A man 6 ft tall walks at a rate of $5 \text{ ft}/\text{sec}$ toward a street light that is 16 ft above the ground. At what rate is the length of his shadow changing when he is 10 ft from the base of the light? (Answer: $-3 \text{ ft}/\text{sec}$)

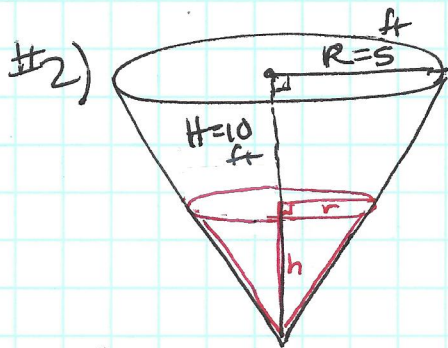
72

Example 4: A pebble is dropped into a calm pond, causing ripples in the form of concentric circles. The radius, r , of the outer ripple is increasing at a constant rate of $1 \text{ ft}/\text{sec}$. When this radius is 4 ft, at what rate is the total area of the disturbed water increasing? (Answer: $8\pi \text{ ft}^2/\text{sec}$)

Example 5: Gravel is falling in a conical pile at the rate of $100 \text{ ft}^3/\text{min}$. Find the rate of change of the height of the pile when the height is 10 ft. Assume that the coarseness of the gravel is such that the radius of the cone is always equal to its height. (Answer: $\frac{1}{\pi} \approx 0.318 \text{ ft}/\text{min}$)

§4.6 EXAMPLE RELATED RATES PROBLEMS.

DAY 71 NOTES



$$\frac{dV}{dt} = 9 \frac{\text{ft}^3}{\text{min}}$$

$$V = \frac{1}{3} \pi r^2 h$$

$$\frac{dh}{dt} = ?$$

When $h = 6 \text{ ft}$

$$\frac{dV}{dt} = \frac{\pi}{3} \left(2rh \frac{dr}{dt} + r^2 \frac{dh}{dt} \right)$$

$$\frac{dV}{dt} = \frac{\pi}{3} \left(2rh \left(\frac{1}{2} \frac{dh}{dt} \right) + r^2 \frac{dh}{dt} \right)$$

$$\frac{dV}{dt} = \frac{\pi}{3} \left((rh + r^2) \frac{dh}{dt} \right)$$

$$\frac{dh}{dt} = \frac{3 \cdot \left(\frac{dV}{dt} \right)}{\pi \cdot (rh + r^2)}$$

$$\frac{dh}{dt} = \frac{3(9)}{\pi(3(6) + 3^2)} = \frac{27}{\pi 27}$$

$$\frac{dh}{dt} = \frac{3 \cdot 9}{\pi(3)(6+3)} = \frac{1}{\pi}$$

ATQ: The water level is rising at a rate of $\frac{1}{\pi} \frac{\text{ft}}{\text{min}}$ when the water is 6ft deep.

SIMILAR TRIANGLES

$$\frac{R}{H} = \frac{r}{h}$$

$$\frac{5}{10} = \frac{r}{h}$$

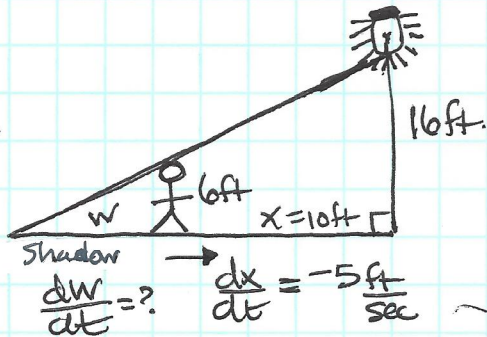
$$h = 6 \therefore r = 3$$

$$\frac{1}{2} h = r$$

$$\therefore \frac{1}{2} \frac{dh}{dt} = \frac{dr}{dt} \leftarrow \text{use this one}$$

$$\text{or } \frac{dh}{dt} = 2 \frac{dr}{dt}$$

#3)



When $x = 10 \text{ ft}$

$$\frac{6}{w} = \frac{16}{w+10}$$

$$6w + 60 = 16w$$

$$60 = 10w$$

$$w = 6 \text{ ft.}$$

$$\frac{w}{6} = \frac{w+x}{16}$$

$$16w = 6w + 6x$$

$$10w = 6x$$

$$w = \frac{3}{5}x$$

$$\frac{dw}{dt} = \frac{3}{5} \frac{dx}{dt}$$

$$\frac{dw}{dt} = \frac{3}{5} \left(-5 \frac{\text{ft}}{\text{sec}} \right)$$

$$\frac{dw}{dt} = -3 \frac{\text{ft}}{\text{sec}}$$

ATQ: The length of the mans shadow is decreasing at a rate of $3 \frac{\text{ft}}{\text{sec}}$ when the man is 10 ft from the lamppost.