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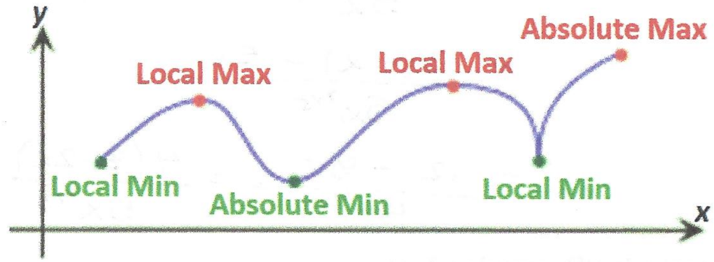
§ 4.1 & § 4.2—Student Notes—Using the First and Second Derivatives

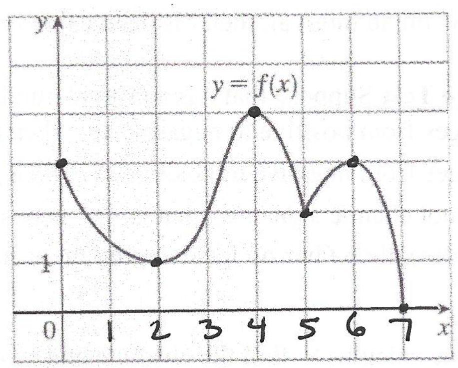
Definition A function f has an **absolute maximum** (or **global maximum**) at c if $f(c) \geq f(x)$ for all x in D, where D is the domain of f. The number $f(c)$ is called the **maximum value** of f on D. Similarly, the function f has an **absolute minimum** (or **global minimum**) at c if $f(c) \leq f(x)$ for all x in D and the number $f(c)$ is called the **minimum value** of f on D. The maximum and minimum values of f are called the **extreme values** of f.

Definition A function f has an **local maximum** (or **relative maximum**) at c if $f(c) \geq f(x)$ when x is near c. [This means that $f(c) \geq f(x)$ for all x in some open interval containing c.] Similarly, the function f has an **local minimum** at c if $f(c) \leq f(x)$ when x is near c.



Example 1: Use the graph to state the absolute and local max/min values

ABS. MAX $f(4) = 4$
ABS MIN. $f(7) = 0$
REL MAX $f(4) = 4$, $f(6) = 3$
REL MIN $f(2) = 1$, $f(5) = 2$



$f(x)$

Example 2: Describe the maximum and minimum, local and absolute, for the following functions:

a. $f(x) = \cos x$

LOCAL & ABS MAX of 1.
infinitely many times at $x = 2\pi k$
LOCAL & ABS MIN of -1
infinitely many times at $x \in \{\pi\} + 2\pi k$

b. $f(x) = x^2$

LOCAL & ABS MIN $f(0) = 0$
NO MAX.

c. $f(x) = x^3$

NO MAX & NO MIN
TERRACE POINT @ $(0, f(0)) = (0,0)$

d. $f(x) = |x|$

LOCAL & ABS MIN $f(0) = 0$
NO MAX.

Definition A critical number of a function f is a number c in the domain of f such that either $f'(c) = 0$ or $f'(c)$ does not exist (DNE).

Theorem If f has a local maximum or minimum at c, and if $f'(c)$ exists, then $f'(c) = 0$.

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Example 3: Find the critical numbers of $f(x)=x^{\frac{3}{5}}(4-x)$.

$$f'(x)=\frac{3}{5}x^{-2/5}(4-x)+x^{3/5}(-1)$$

$$=\frac{3(4-x)}{5x^{2/5}}+\frac{-x^{3/5}}{1}$$

$$=\frac{3(4-x)-5x}{5x^{2/5}}$$

$$=\frac{12-8x}{5x^{2/5}}=\frac{4(3-2x)}{5x^{2/5}}$$

CRITICAL POINTS

- $f'(x)=0$ when
  $3-2x=0$
  $x=\frac{3}{2}$
- $f'(x)$ undefined when
  $5x^{2/5}=0$
  $x=0$

$\therefore$ $\boxed{\text{C.P. } x=0,\ \frac{3}{2}}$

Increasing/Decreasing Test
(a) If $f'(x)>0$ on an interval, then *f* is increasing on that interval
(b) If $f'(x)<0$ on an interval, then *f* is decreasing on that interval

First Derivative Test Suppose that *c* is a critical number of a continuous function *f*.
(a) If f' changes from positive to negative at *c*, then *f* has a local maximum at *c*.
(b) If f' changes from negative to positive at *c*, then *f* has a local minimum at *c*.
(c) If f' does not change sign at *c*, (that is, f' is positive on both sides of *c* or negative on both sides), then *f* has no local maximum or minimum at *c*.

Example 4: Use calculus to find the absolute and relative minimum and maximum values of the function $f(x)=\frac{\ln x}{x}$, *on* $\boxed{[1,3]}$ then check your results using your calculator.
← restricted domain

$$f'(x)=\frac{x\frac{1}{x}-\ln x}{x^2}$$

$$f'(x)=\frac{1-\ln x}{x^2}$$

CRITICAL POINTS
$f'(x)=0 \rightarrow x=e$
$f'(x)$ und $\rightarrow x=0$

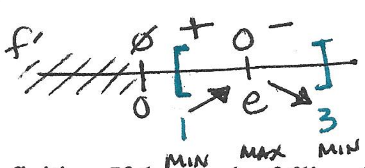

TABLE ⟶ CLASSIFY

x	$f(x)$	
1	0	$f(1)=0$ is ABS. MIN
e	$\frac{1}{e}\approx 0.368$	$f(e)=\frac{1}{e}$ is ABS. MAX. REL MAX
3	$\frac{1}{3}\ln 3\approx 0.366$	$f(3)=\frac{\ln 3}{3}$ is REL MIN

Definition If the graph of *f* lies above all of its tangents on an interval *I*, then it is called **concave upward** on *I*. If the graph of *f* lies below all of its tangents on an interval *I*, then it is called **concave downward** on *I*.

Concavity Test
(a) If $f''(x)>0$ for all *x* on *I*, then the graph of *f* is concave upward on *I*.
(b) If $f''(x)<0$ for all *x* on *I*, then the graph of *f* is concave downward on *I*.

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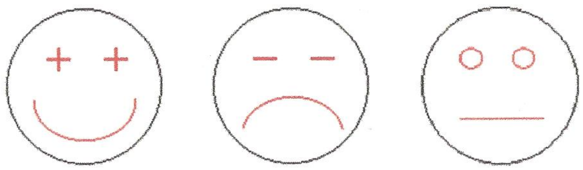
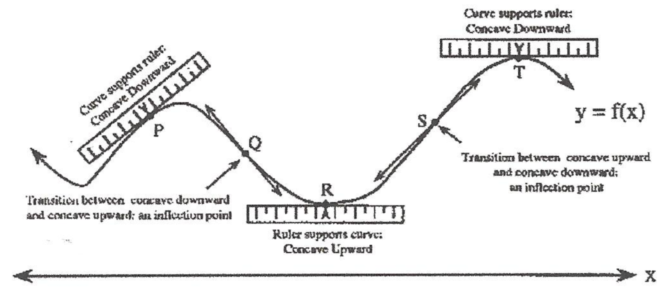
Test for Concavity:

A function --- is concave up when f "(x) > 0
--- is concave down when f "(x) < 0
--- has no concavity when f "(x) = 0
--- may have a possible point of inflection if f "(x) = 0.
--- will have a point of inflection if f "(x) = 0 and changes signs.





Second Derivative Test Suppose f'' is continuous near c.
(a) If $f'(c)=0$ and $f''(c)>0$, then f has a local minimum at c.
(b) If $f'(c)=0$ and $f''(c)<0$, then f has a local maximum at c.

Example 5. Given $g(x)=x+2\sin x \quad 0\le x\le 2\pi$, use the second Derivative Test to find the relative extrema and then find the intervals concavity, points of inflection, and use the information to sketch the curve.

$g'(x) = 1+2\cos x$

$g'(x)=0 \quad \cos x = -\frac{1}{2}$

$x \in \{\frac{2\pi}{3}, \frac{4\pi}{3}\} + 2\pi k$

$g'(x)$: + 0 − 0 +
0 ↗ $\frac{2\pi}{3}$ ↘ $\frac{4\pi}{3}$ ↗ 2π
MIN MAX MIN MAX

$g''(x) = -2\sin x$

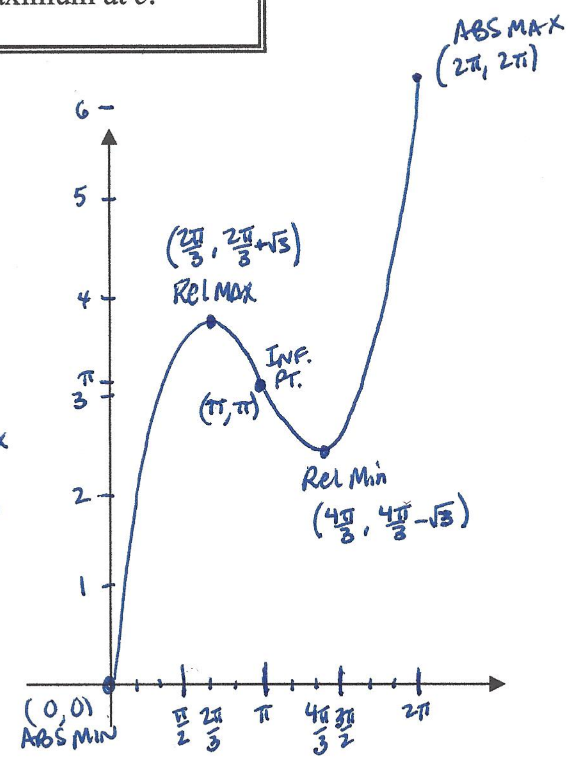
$g''(x)=0 \quad x\in\{0,\pi\}+2\pi k$

$g''(x)$: − +
0 ⌒ π ⌣ 2π

TABLE

x	$g(x)$
0	0 ABS MIN
$\frac{2\pi}{3}$	$\frac{2\pi}{3}+\sqrt{3} \approx 3.826$ Rel MAX
$\frac{4\pi}{3}$	$\frac{4\pi}{3}-\sqrt{3} \approx 2.457$ Rel MIN
2π	$2\pi \approx 6.283$ ABS MAX.

Inflection Point: (π,π) b/c g'' changes sign ⊖ to ⊕ at $x=\pi$.



Example 6: Given $g(x)=x+2\sin x \quad 0\le x\le 2\pi$ find the intervals concavity, points of inflection, and use the intervals of increasing/decreasing and local maxima and minima to sketch the curve.

$g(x)$ is increasing on $(0,\frac{2\pi}{3}),(\frac{4\pi}{3},2\pi)$ b/c $g'(x)>0$.
decreasing on $(\frac{2\pi}{3},\frac{4\pi}{3})$ b/c $g'(x)<0$.
concave up on $(\pi,2\pi)$ b/c $g''(x)>0$.
concave down on $(0,\pi)$ b/c $g''(x)<0$.

2nd DERIVATIVE TEST to JUSTIFY EXTREMA → [Evaluate $g''(x)$ for critical points & make conclusion].

$g''(\frac{2\pi}{3}) = -2\sin(\frac{2\pi}{3}) = -\sqrt{3} < 0$. Since $g''(\frac{2\pi}{3})<0$ g is concave down at C.P. $x=\frac{2\pi}{3}$ therefore $g(\frac{2\pi}{3})$ is a MAXIMUM.

$g''(\frac{4\pi}{3}) = -2\sin(\frac{4\pi}{3}) = +\sqrt{3} > 0$. Since $g''(\frac{4\pi}{3})>0$ g is concave up at C.P. $x=\frac{4\pi}{3}$ therefore $g(\frac{4\pi}{3})$ is a MINIMUM.