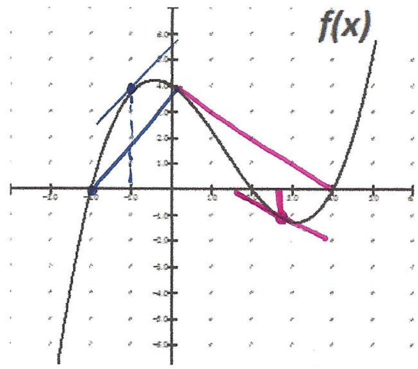
DAY 58 CLASS NOTES

§3.10 Mean Value Theorem – Student Notes

MEAN VALUE THEOREM: **If a function is <u>continuous</u> on [a, b] and <u>differentiable</u> on (a, b), then there is as number c in the interval (a, b) such that**

$$f'(c) = \frac{f(b)-f(a)}{b-a} \quad \text{or} \quad (b-a)\ f'(c) = f(b) - f(a).$$

1. Use the graph to illustrate the Mean Value Theorem with a continuous and differentiable function. Show *f (x), a, b, c* and all other conditions of the theorem.



EX. $x \in [-2, 0]$

$$\frac{f(0)-f(-2)}{0+2} = f'(-1)$$

EX. $x \in [0, 4]$

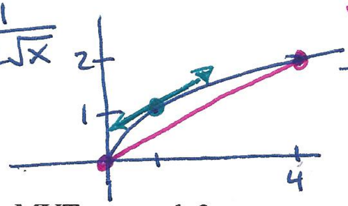
$$\frac{f(4)-f(0)}{4-0} = f'(2.8)$$

2. Find the number c that satisfies the Mean Value Theorem (MVT) for f (x) = $\sqrt{x}$ on the interval [0,4]. Draw a picture.

f(x) is continuous on the closed interval [0,4] and differentiable on the open (0,4)

$f(x) = \sqrt{x}$

$f'(x) = \frac{1}{2\sqrt{x}}$



$$\boxed{\frac{f(4)-f(0)}{4-0}} = \frac{2-0}{4} = \boxed{\frac{1}{2}}$$

$\boxed{f'(1) = \frac{1}{2}}$

$\frac{1}{2\sqrt{x}} = \frac{1}{2}$? →

$$\boxed{\therefore f'(1) = \frac{f(4)-f(0)}{4-0} \quad c = 1}$$

3. Why does the MVT not apply?

a) $y = \frac{x+3}{x-2}$ on [0,3]

The function is not continuous at $x = 2$ ∴ the MVT does not apply.

b) $f(x) = x^{\frac{1}{3}}$ on [-1,1] → $f'(x) = \frac{1}{3\sqrt[3]{x^2}}$

f(x) is continuous on [-1,1] but f(x) is not differentiable on (-1,1) b/c f'(0) is undefined ∴ the MVT does not apply.

4. Apply the MVT, if possible. If not possible explain why.

A $f(x) = x^2$ on [-2,1]	B $f(x) = x^3 - 3x^2$ on [0,3]	C $f(x) = x^{\frac{2}{3}}$ on [0,1]
✓ f(x) is continuous on [-2,1] ✓ f(x) is differentiable on (-2,1) $f'(x) = 2x = \frac{f(1)-f(-2)}{1+2}$ $2x = \frac{1-4}{1+2}$ $2x = \frac{-3}{3}$ ∴ $x = \frac{1}{2}$ $f'(-\frac{1}{2}) = \frac{f(1)-f(-2)}{1+2} = -1$ ∴ MVT is satisfied for f(x) on [-2,1] when $c = -\frac{1}{2}$.	✓ f(x) is continuous on [0,3] ✓ f'(x) is diff on (0,3) $f'(x) = 3x^2 - 6x$ $3x^2 - 6x = \frac{f(3)-f(0)}{3-0} = 0$ $3x(x-2) = 0$ $x = 0$ & $x = 2$ $f'(0) = f'(2) = \frac{f(3)-f(0)}{3-0} = 0$ ∴ The MVT is satisfied for f(x) on [0,3] when c = 0 & 2.	✓ f(x) is continuous on [0,1] ✓ f'(x) is diff on (0,1) $f'(x) = \frac{2}{3}x^{-\frac{1}{3}} = \frac{2}{3\sqrt[3]{x}}$ $\frac{2}{3\sqrt[3]{x}} = \frac{f(1)-f(0)}{1-0} = 1$ $\sqrt[3]{x} = \frac{2}{3}$ ∴ $x = \frac{8}{27}$ $f'(\frac{8}{27}) = \frac{f(1)-f(0)}{1-0} = 1$ ∴ MVT is satisfied for f(x) on [0,1] when $c = \frac{8}{27}$.

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MVT Problems

1. The function $f(x) = x^{\frac{2}{3}}$ on [- 8, 8] does not satisfy the conditions of the Mean Value Theorem because

   A. $f(0)$ is not defined B. $f(x)$ is not continuous of [-8, 8]

   C. $f'(-1)$ does not exist D. $f(x)$ is not defined for $x < 0$.

   E. $f'(0)$ does not exist

   f(x) is continuous on [-8, 8] but f(x) is not differentiable at $x=0$ ∴ $f'(0)$ DNE (sharp point)

2. If $f(a) = f(b)$ and $f(x)$ is continuous on [a, b], then

   ~~A.~~ $f(x)$ must be identically zero

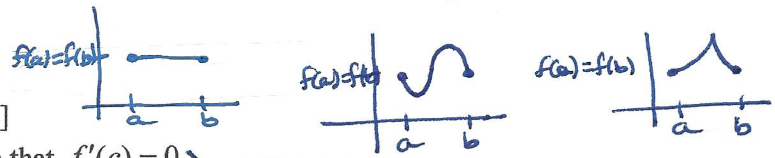
   ~~B.~~ $f'(x)$ may be different from zero for all x on [a,b]

   ~~C.~~ there exists at least one number c, a < c< b, such that $f'(c) = 0$

   ~~D.~~ $f'(x)$ must exist for every x on (a, b)

   E. none of the preceding is true

   only if f(x) is differentiable on (a, b) & MVT applies



3. Find the value of c that satisfies the Mean Value Theorem for $f(x) = x^3 + x - 4$ on the interval [- 2, 1].

   f(x) is continuous on [-2, 1] & differentiable on (-2, 1)

   A. – 1 B. 1 C. 0 D. 4 E. None of these.

   $f'(c) = \frac{f(1)-f(-2)}{1+2}$

   $3c^2+1 = \frac{-2+14}{3}$

   $3c^2+1 = 4$

   $3c^2 = 3$

   $c^2 = 1$

   $c = 1$ or $c = -1$

4. Find the number that satisfies the MVT on the given interval <u>**or**</u> state why the theorem does not apply.

   a) $f(x) = x^{\frac{2}{5}}$ on [0, 32]

   Sharp pt. @ $x=0$

   f(x) is cont. on [0, 32]

   f(x) is diff. on (0, 32)

   $f'(c) = \frac{2}{5}c^{-3/5} = \frac{2}{5c^{3/5}} = \frac{f(32)-f(0)}{32}$

   $\frac{4-0}{32} = \frac{1}{8} = \frac{2}{5c^{3/5}} \rightarrow c^{3/5} = 16$

   $\boxed{c = 16^{5/3}}$ by MVT.

   b) $f(x) = \frac{1}{(x-2)^2}$ on [2, 5]

   f(x) is discontinuous at $x=2$ ∴ MVT does not apply

   c) $g(x) = x + \frac{1}{x}$ on [1, 3]

   g(x) is continuous on [1, 3] & diff on (1, 3) ∴

   $g'(c) = 1 - \frac{1}{c^2} = \frac{c^2-1}{c^2} = \frac{g(3)-g(1)}{3-1}$

   $\frac{c^2-1}{c^2} = \frac{\frac{4}{3}-2}{2} = \frac{2}{3} - 1 = \frac{-1}{3}$

   $\frac{c^2-1}{c^2} = \frac{-1}{3} \rightarrow 3c^2 - 3 = -c^2$

   $4c^2 = 3$ $c = \pm\frac{\sqrt{3}}{2}$ ∴ $\boxed{c = +\frac{\sqrt{3}}{2}}$ by MVT.

   d) $h(x) = x^{\frac{1}{2}} + 2(x-2)^{\frac{1}{3}}$ on [1, 9]

   h(x) is not differentiable at $x=2$ b/c $\sqrt[3]{x-2}$ has vertical tangent.

   ∴ MVT does not apply.

2003 #92: Let f be defined by $f(x) = x + \ln(x)$. What is the value of c for which the instantaneous rate of change of f at $x = c$ is the same as the average rate of change of f over $[1,4]$?

(*A*) 0.456 (*B*) 1.244 (*C*) 2.164 (*D*) 2.342 (*E*) 2.452

$f(x) = x + \ln(x)$ is continuous on [1, 4] & differentiable on (1, 4).

$f'(x) = 1 + \frac{1}{x} = \frac{x+1}{x} = \frac{f(4)-f(1)}{3} =$

$1 + \frac{1}{x} = \frac{x+1}{x} = \frac{(4+\ln 4) - (1+\ln 1)}{3} = \frac{3+\ln 4}{3} = \frac{x+1}{x}$

or $1 + \frac{1}{x} = 1 + \frac{\ln 4}{3} \rightarrow x = \frac{3}{\ln 4}$

$\frac{3+\ln 4}{3} = \frac{x+1}{x}$

$(3+\ln 4)x = 3x + 3$

$(3+\ln 4)x - 3x = 3$

$x(3+\ln 4 - 3) = 3$

$x(\ln 4) = 3$

$x = \frac{3}{\ln 4}$

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