DAY 56 NOTES.

§3.7 Implicit Differentiation -- Student Notes

Up to now, we have worked explicitly, solving an equation for one variable in terms of another. For example, if you were asked to find $\frac{dy}{dx}$ for $2x^2 + y^2 = 4$, you would solve for y and get $y = \pm\sqrt{4-2x^2}$ and then take the derivative. Sometimes it is inconvenient or difficult to solve for y. In this case, we use implicit differentiation. You assume y could be solved in terms of x and treat it as a function in terms of x. Thus, you must apply the chain rule because you are assuming y is defined in terms of x.

Differentiating with respect to x:

$$\frac{d}{dx}\left[x^3\right] = 3x^2$$

Variables agree $\Rightarrow$ use power rule

variables agree

$$\frac{d}{dx}\left[y^3\right] = 3y^2\frac{dy}{dx}$$

Variables disagree $\Rightarrow$ use power rule and chain rule

variables disagree

$$\frac{d}{dx}\left[x+3y\right] = 1 + 3\frac{dy}{dx}$$

variables disagree

variables agree

$$\frac{d}{dx}\left[xy^2\right] = 1\left(y^2\right) + x\left(2y\frac{dy}{dx}\right) = y^2 + 2xy\frac{dy}{dx}$$

variables disagree

use product and chain rules

Consider the problem, find $\frac{dy}{dx}$ for $y^2 - 2y + 3x = x^2$. Treat y as a quantity in terms of x so

$$\frac{d}{dx}\left(y^2\right) - \frac{d}{dx}\left(2y\right) + \frac{d}{dx}\left(3x\right) = \frac{d}{dx}\left(x^2\right)$$

Different Same

$$2y\frac{dy}{dx} \qquad -2\frac{dy}{dx} \qquad +3 \qquad = 2x$$

Now solve for $\frac{dy}{dx}$.

$$2y\frac{dy}{dx} - 2\frac{dy}{dx} = 2x - 3 \Rightarrow \frac{dy}{dx}\left(2y-2\right) = 2x-3 \Rightarrow \frac{dy}{dx} = \frac{2x-3}{2y-2}$$

§3.7 CLASS NOTES. DAY 56

Guidelines for Implicit Differentiation:

1. Differentiate both sides of the equation with respect to x.
2. Collect all terms involving $\frac{dy}{dx}$ on one side of the equation and move all other terms to the other side.
3. Factor $\frac{dy}{dx}$ out of the terms on the one side.
4. Solve for $\frac{dy}{dx}$ by dividing both sides of the equation by the factored term.

Practice:

Find $\frac{dy}{dx}$:

1. $y^3 + 7\cos(y) = x^3$

$$3y^2\frac{dy}{dx} - 7\sin(y)\frac{dy}{dx} = 3x^2$$
$$\frac{dy}{dx}\left[3y^2 - 7\sin(y)\right] = 3x^2$$
$$\frac{dy}{dx} = \frac{3x^2}{[3y^2 - 7\sin(y)]}$$

product Rule

$d[x^2y] = 2xy + x^2\frac{dy}{dx}$

2. $4x^2y - 3y = x^3 - 1$

$$4\left[2xy + x^2\frac{dy}{dx}\right] - 3\frac{dy}{dx} = 3x^2$$
$$(4x^2-3)\frac{dy}{dx} = 3x^2 - 8xy \rightarrow \frac{dy}{dx} = \frac{3x^2-8xy}{(4x^2-3)}$$

3. $x^2 + 5y^3 = xy + 9$

$$2x + 15y^2\frac{dy}{dx} = y + x\frac{dy}{dx}$$
$$(15y^2 - x)\frac{dy}{dx} = y - 2x \longrightarrow \frac{dy}{dx} = \frac{(y-2x)}{(15y^2-x)}$$

4. Find $\frac{dy}{dt}$ if $t^3 + t^2y - 10y^4 = 0$

$$3t^2 + 2ty + t^2\frac{dy}{dx} - 40y^3\frac{dy}{dx} = 0$$
$$(t^2 - 40y^3)\frac{dy}{dx} = -(3t^2 + 2ty)$$
$$\frac{dy}{dx} = \frac{3t^2+2ty}{40y^3 - t^2}$$

slopes of $\perp$ lines m & $\frac{-1}{m}$

5. Find the equation of the normal line (the line perpendicular to the tangent line) to the curve $\cancel{8}(x^2+y^2)^2 = \cancel{100}(x^2-y^2)$ at the point (3, 1).

tangent line $\boxed{y = -\frac{13}{9}(x-3)+1} \rightarrow \boxed{y = \frac{9}{13}(x-3)+1 \quad \text{NORMAL LINE}}$

$$\frac{d}{dx}\left[2(x^2+y^2)^2 = 25(x^2-y^2)\right]$$
$$4(x^2+y^2)\cdot\left(2x + 2y\frac{dy}{dx}\right) = 25\left(2x - 2y\frac{dy}{dx}\right)$$
$$4(x^2+y^2)(2x) - 25(2x) = \left[25(-2y) - 4(x^2+y^2)(2y)\right]\frac{dy}{dx}$$
$$(2x)\left(4(x^2+y^2) - 25\right) = \left[(-2y)\left(25 + 4(x^2+y^2)\right)\right]\frac{dy}{dx}$$

@ (1,3)

$$\frac{dy}{dx} = \frac{(-\cancel{2}y)\left[4(x^2+y^2)+25\right]}{(\cancel{2}x)\left[4(x^2+y^2)-25\right]}\Bigg|_{(3,1)} = \frac{(-1)[40+25]}{3[40-25]} = \frac{-1[8+5]}{3[8-5]} = \boxed{\frac{-13}{9} = \frac{dy}{dx}}$$