Chapter 11 Packet

11.1 & 11.2 What is a Differential Equation and What are Slope Fields

What is a differential equation? An equation that gives information about the rate of change of an unknown function is called a differential equation; ie, it is an equation with a derivative in it. For example,

$$\frac{dy}{dx} = -\frac{x}{y} \qquad \frac{dy}{dt} = 100 - y \qquad x\frac{dy}{dx} - 3y = 0 \qquad x^2\frac{d^2y}{dx^2} = 2$$

A function that satisfies the differential equation is called a solution.

Example: Is $y = e^{-2t}$ a solution to $\frac{d^2y}{dt^2} - 2\frac{dy}{dt} - 8y = 0$?

There are many types of differential equations. In this course, we will solve separable, first order differential equations.

Definition A separable equation is a first-order differential equation in which the expression for $\frac{dy}{dx}$ can be factored as a function of *x* times a function of *y*. In other words, it can be written in the form

$$\frac{dy}{dx} = g(x)f(y)$$

The name separable comes from the fact that the expression on the right side can be "separated" into a function of x and a function of *y*. Equivalently, we could write

$$\frac{dy}{dx} = \frac{g(x)}{h(y)} \quad \text{or} \quad h(y)dy = g(x)dx$$

so that all y's are on one side of the equation and all x's are on the other side. The differential can't end up in the denominator though!

Then we integrate both sides of the equation:

$$\int h(y)dy = \int g(x)dx$$

Are They Separable?

You see below ten differential equations. Some of them are separable, and some are not. You do *not* have to solve any of them. For each one, classify it as separable or inseparable. If it is separable, separate it.

Example: $y' = xy$

Answer: Separable. $\frac{dy}{y} = x\,dx$

---

1. $y' = \frac{x}{\sqrt{y}}$

2. $y' = x + y$

3. $y' = \ln(xy)$

4. $y' = \frac{\ln x + x}{\ln y + y}$

5. $y' = ye^{\sin x + \cos y}$

6. $y' = \ln(x^y)$

7. $y' = \sin(x^y)$

8. $y' = y \sin x + xy$

9. $y' = \frac{xy + y}{2x - 3xy}$

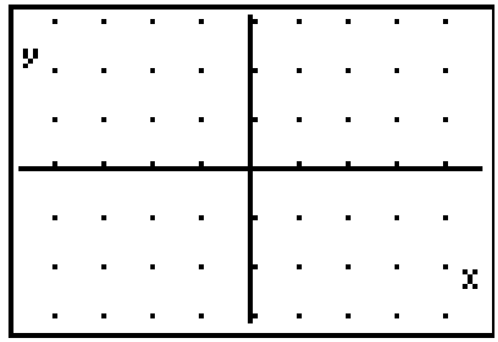
10. $y' = xy - 2x + y - 2$

Differential equations give you information about how the slope of some function family behaves at points on the curve.

- The ***general solution*** is the family of curves.
- A ***particular solution*** satisfies some initial condition.

<u>Slope Card Activity</u>

Problem: Visualize solutions to the differential equation: $\frac{dy}{dx} = \frac{-x}{y}$



1) Using the slope cards, find and draw small line segments for the slope of solution curves at every grid point.

- What do the solution curves to this differential equation look like?

2) Sketch the particular solution for the initial condition, $x = 1$, $y = 0$ by finding the point on the slope field and going in the direction of the tangent line, and repeat…

3) Sketch two more solutions to the differential equation on the slope field.

4) Describe the family of solutions to this differential equation.

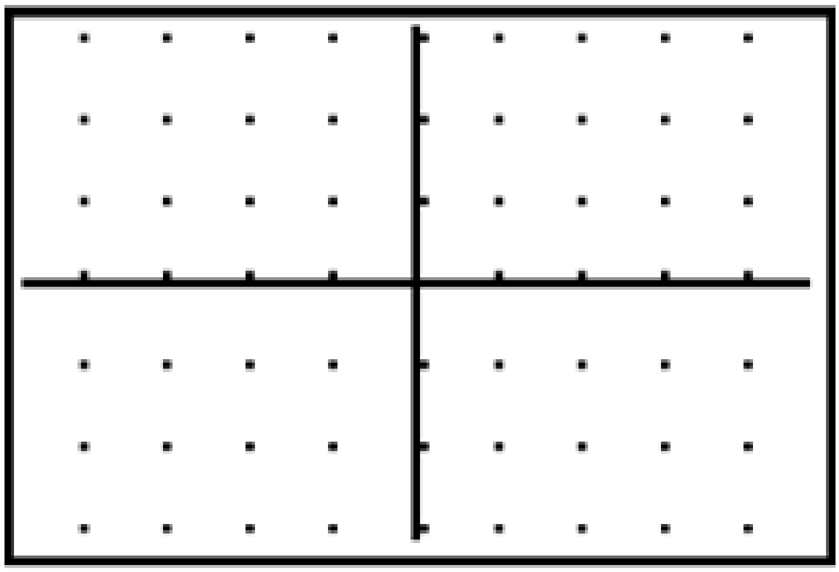
5) Write an equation for this family of solution curves.

6) Differentiate your equation with respect to x.

7) Compare your differential equation to the given one.

We can use graphical representations with slope field (direction field) to identify the unique solution. The best way to understand slope fields is to draw some by hand. To do this, we draw small segments of tangent lines at selected points. We can do this because a differentiable function is locally linear at the point of tangency and can be approximated by its tangent line over small intervals.

1. Given the differential equation $\frac{dy}{dx} = x$, at each grid point, calculate the value of the slope and draw a short line segment representing that slope.

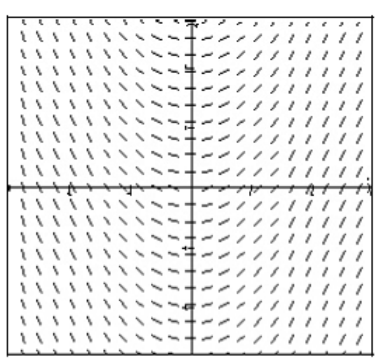


a. Given the initial condition f(0) = 0 sketch the particular solution.

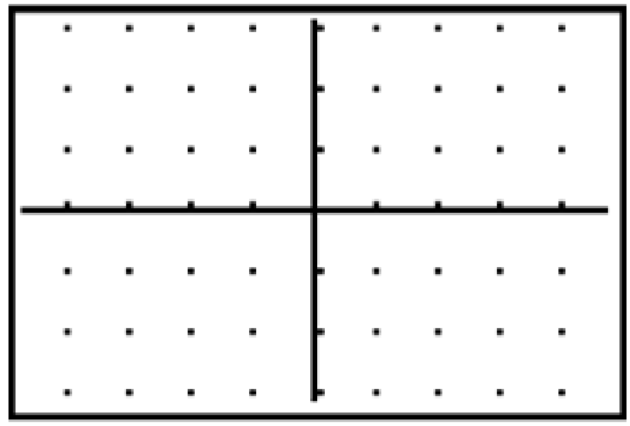
b. Given the initial condition f(0) = -2 sketch the particular solution.

c. What family of functions seems to match the slope field?

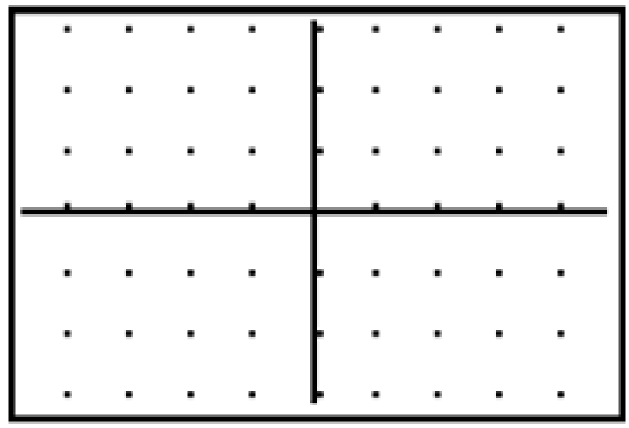
2. This is what the slope field would look like if graphed by a computer/calculator:



3. Given $\dfrac{dy}{dx} = 2x$, sketch the slope field.



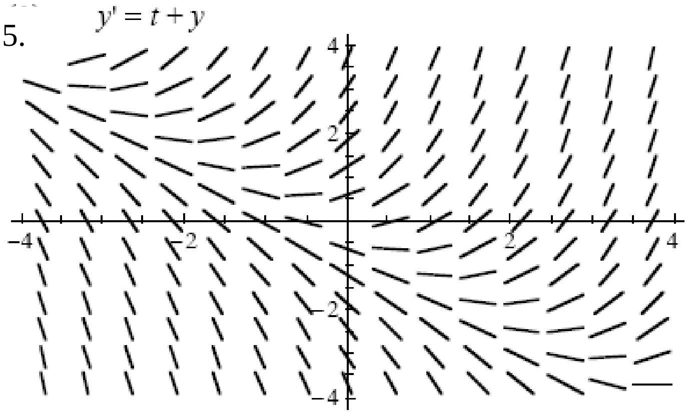
4. Given $\dfrac{dy}{dx} = \dfrac{1}{x+3}$, sketch the slope field.



The solution curves are hiding in the slope field. Given one point of a particular solution curve, you can sketch the graph from that point, in both directions, to see the graph of the solutions

Draw possible solution curves to the differential equations below going through each of the given points. Be sure to label each one.

5. $y' = t + y$



$A(-2.5, 2)$

$B(-3, 2)$

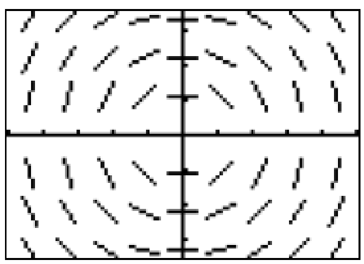
$C(-3.5, 2)$

To match the slope field with a particular equation, follow these guidelines:

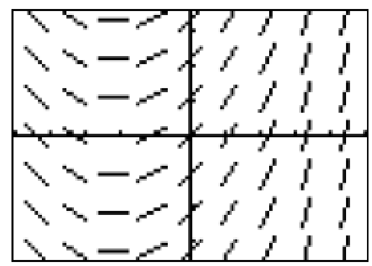
1. If the slopes are the same along vertical lines, then the differential equation is only in terms of _______
2. If the slopes are the same along horizontal lines, then the differential equation is only in terms of _______
3. If the slope field has a combination of slopes horizontally and vertically, then the differential equation is in terms of both ______________
4. Find where the slopes should be 0 or undefined. These should stand out.
5. Think about whether slopes in a single quadrant should all be positive or all negative.
6. If you still can't decide, choose a point, calculate the slope, and see if it matches.
7.

Match the slope fields with their differential equations.

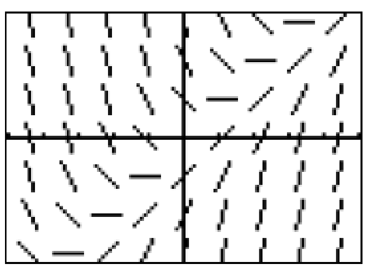
(A)



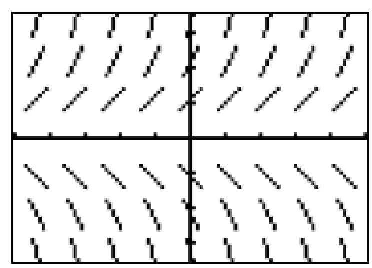
(B)



(C)



(D)



15. $\dfrac{dy}{dx} = \dfrac{1}{2}x + 1$

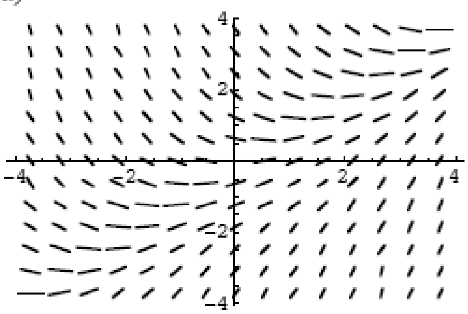
16. $\dfrac{dy}{dx} = y$

17. $\dfrac{dy}{dx} = x - y$

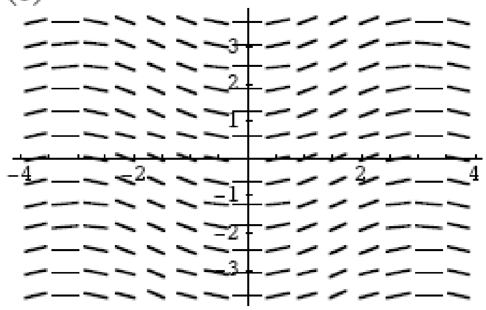
18. $\dfrac{dy}{dx} = -\dfrac{x}{y}$

8. Match each slope field with the correct differential equation.

(a)



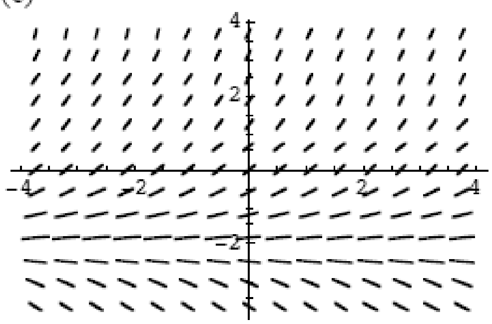
(b)



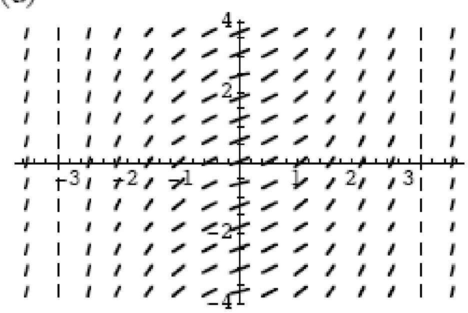
(1) $y' = y + 2$

(2) $y' = x^2 + 1$

(c)



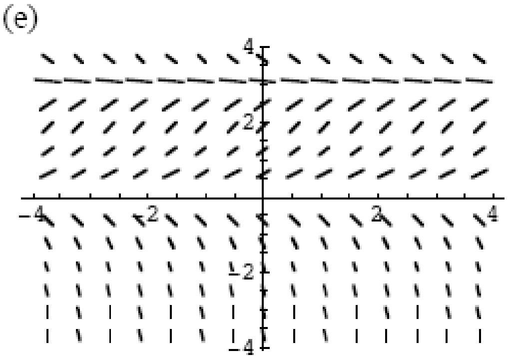
(d)



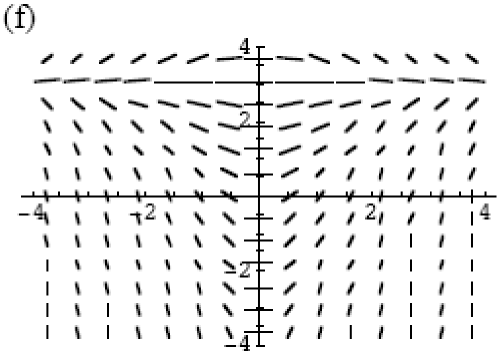
(3) $y' = x\,(3 - y)$

(4) $y' = \sin x$

(e)



(f)



(5) $y' = x - y$

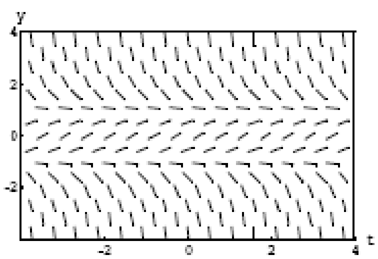
(6) $y' = y\,(3 - y)$

9.

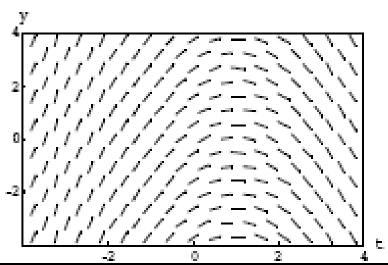
Match the following differential equations to their slope fields. Also, on each slope field, sketch the solution curve satisfying the initial condition $y(0)=1$ for as much of the interval $-4<t<4$ as fits on the slope field.

____ $\frac{dy}{dt}=t-1$	____ $\frac{dy}{dt}=1-y^2$	____ $\frac{dy}{dt}=y^2-t^2$
____ $\frac{dy}{dt}=1-t$	____ $\frac{dy}{dt}=1-y$	____ $\frac{dy}{dt}=t^2-y^2$
____ $\frac{dy}{dt}=1+y$	____ $\frac{dy}{dt}=y^2-1$	____ $\frac{dy}{dt}=0.5y(1-\frac{y}{2})$

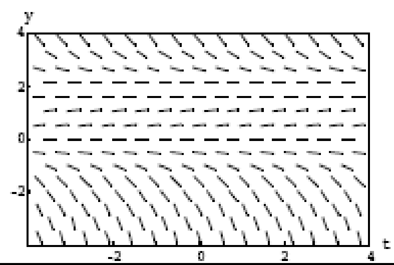
A.



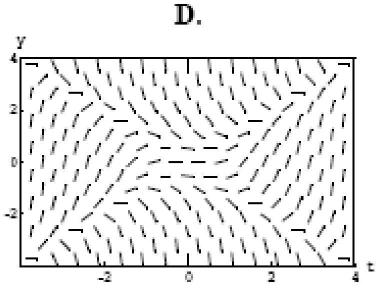
B.



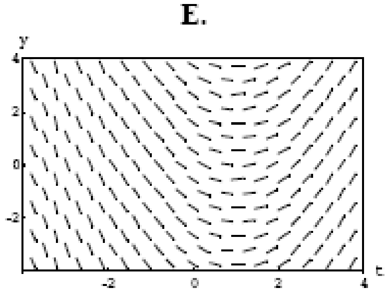
C.



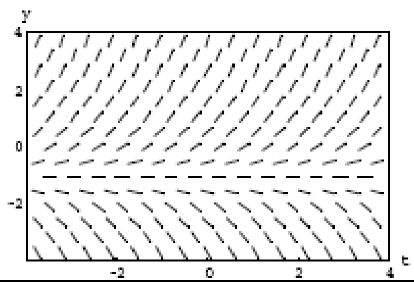
D.



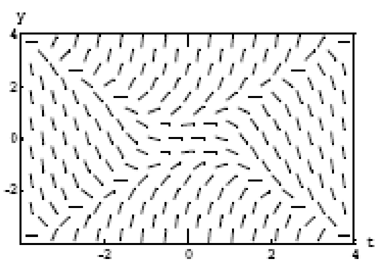
E.



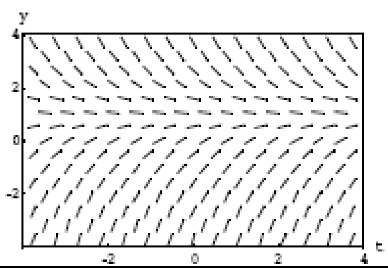
F.



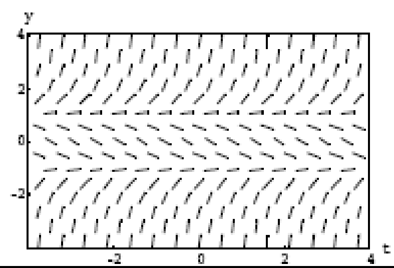
G.



H.



I.



10. The slope field for the differential equation $\frac{dy}{dx} = \frac{x^2 y + y^2}{4x + 2y}$ will have vertical segments when

(A) $y = 2x$, only

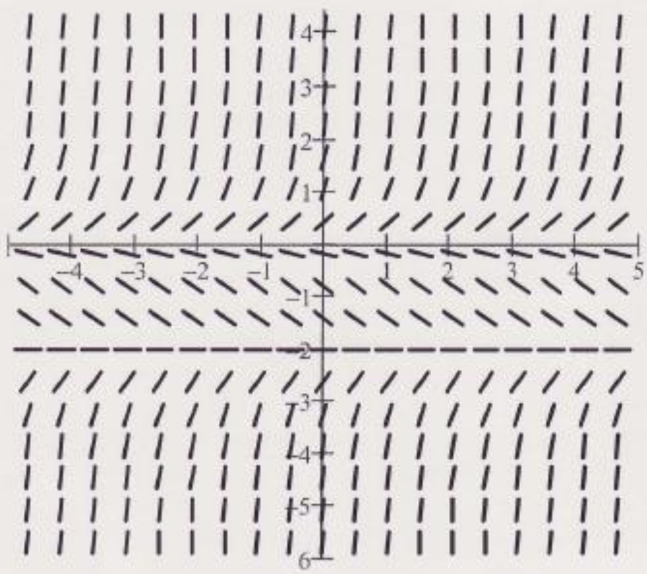
(B) $y = -2x$, only

(C) $y = -x^2$, only

(D) $y = 0$, only

(E) $y = 0$ or $y = -x^2$

11.



Which statement is true about the solutions $y(x)$, of a differential equation whose slope field is shown above?

I. If $y(0) > 0$ then $\lim_{x \to \infty} y(x) \approx 0$.

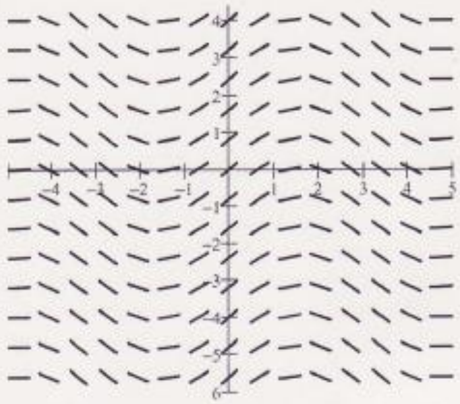
II. If $-2 < y(0) < 0$ then $\lim_{x \to \infty} y(x) \approx -2$.

III. If $y(0) < -2$ then $\lim_{x \to \infty} y(x) \approx -2$.

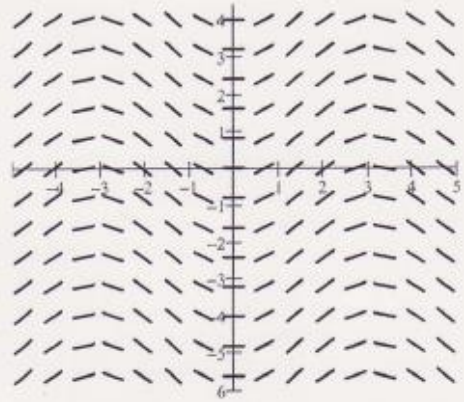
(A) I only (B) II only (C) III only (D) II and III only (E) I, II, and III

12. Which choice represents the slope field for $\frac{dy}{dx} = \cos x$?

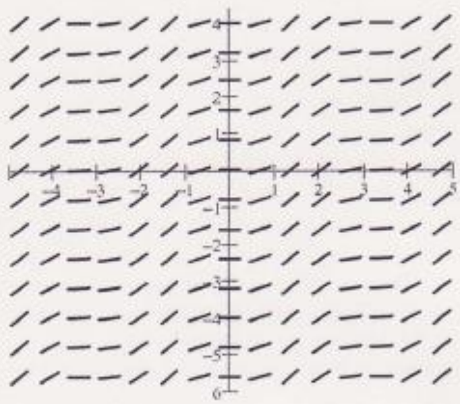
(A)



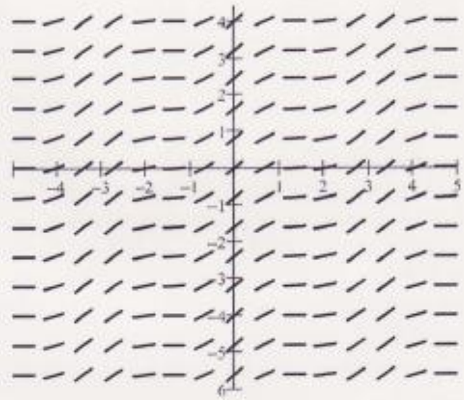
(B)



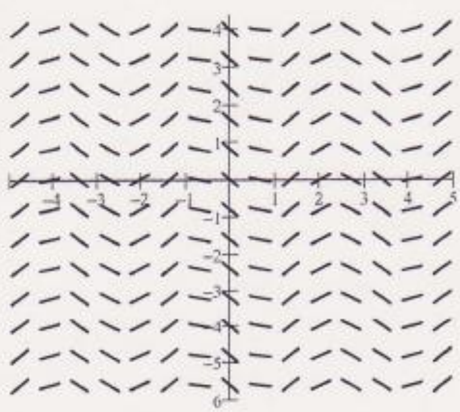
(C)



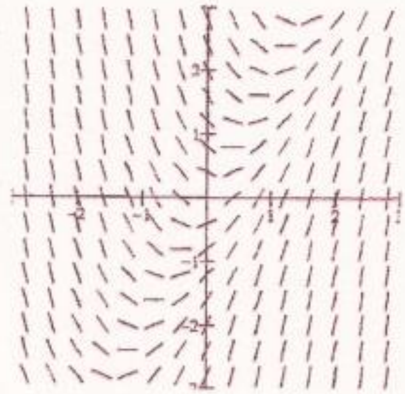
(D)



(E)

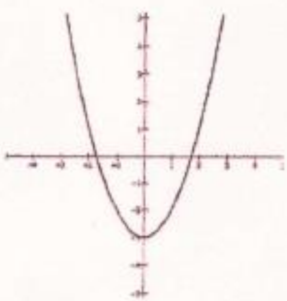


13. Slope field $y' = 2x - y$

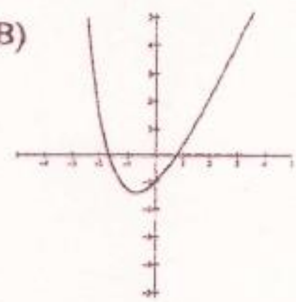


Which graph could be a solution of the differential equation that is shown above?

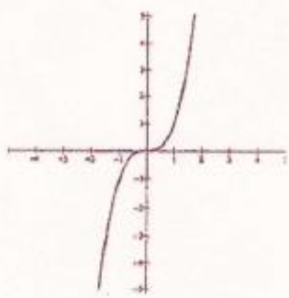
(A)



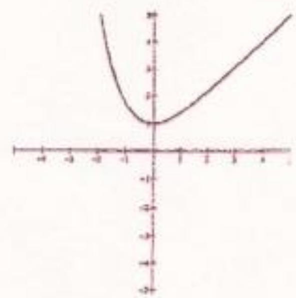
(B)



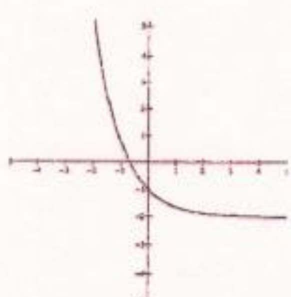
(C)



(D)

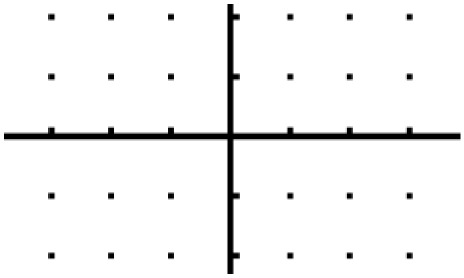


(E)

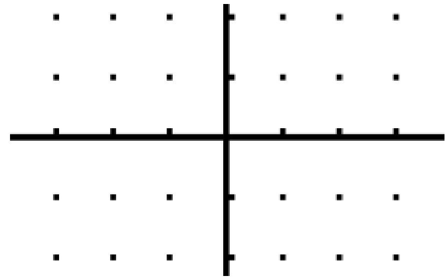


Draw a slope field for each of the following differential equations.

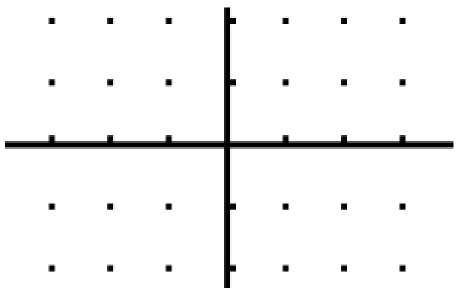
1. $\frac{dy}{dx} = x + 1$



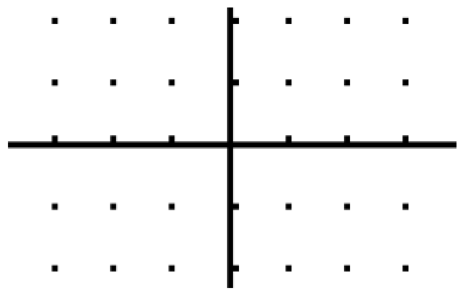
2. $\frac{dy}{dx} = 2y$



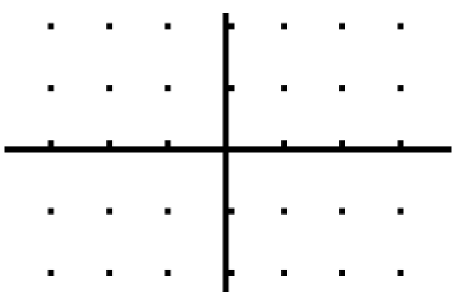
3. $\frac{dy}{dx} = x + y$



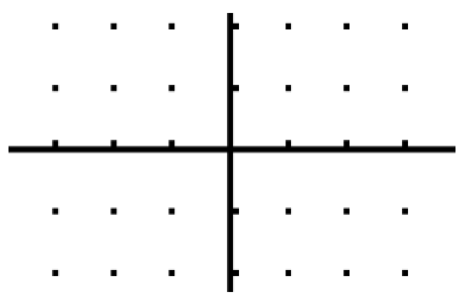
4. $\frac{dy}{dx} = 2x$



5. $\frac{dy}{dx} = y - 1$



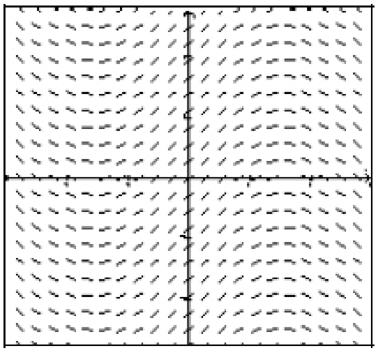
6. $\frac{dy}{dx} = -\frac{y}{x}$



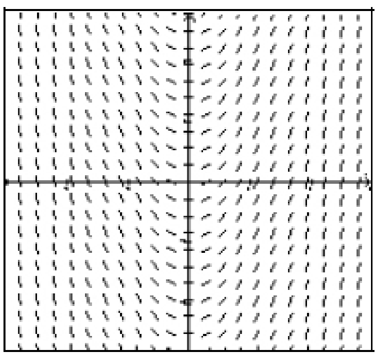
7.

Below are six examples of **slope fields**. Match them with the correct differential equation. Explain each choice.

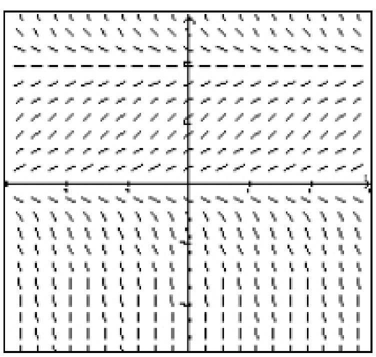
a.



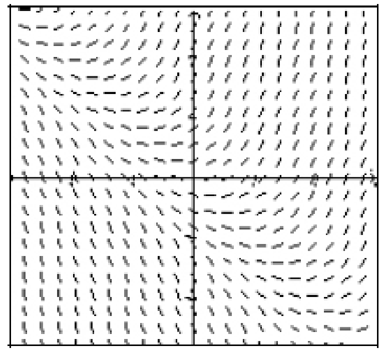
b.



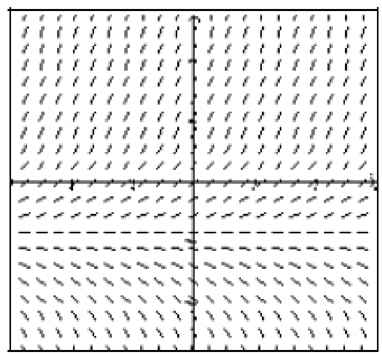
c.



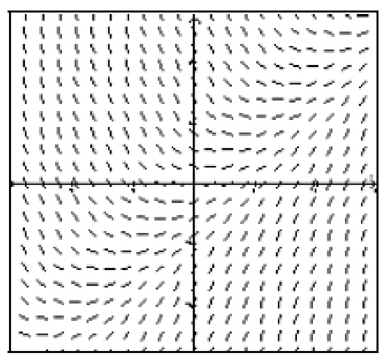
d.



e.



f.



1. $\frac{dy}{dx} = x - y$

2. $\frac{dy}{dx} = 1 + y$

3. $\frac{dy}{dx} = \cos x$

4. $\frac{dy}{dx} = 2x$

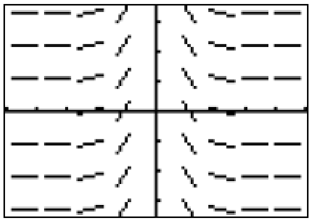
5. $\frac{dy}{dx} = x + y$

6. $\frac{dy}{dx} = y(2 - y)$

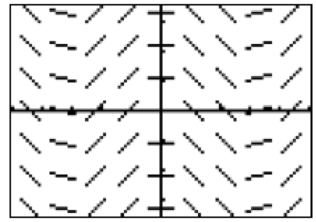
8.

Match each slope field with the equation that the slope field could represent.

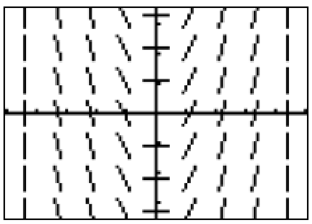
(A)



(B)

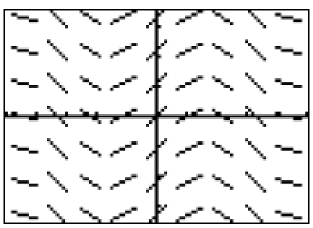


(C)

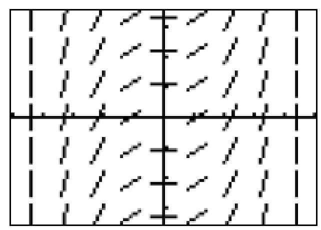


(D)

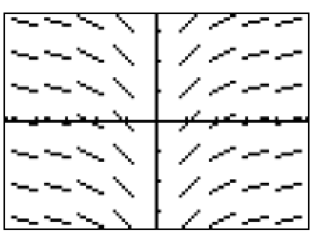
(E)



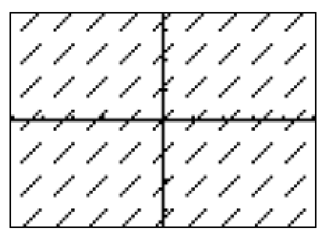
(F)



(G)



(H)



7. $y = 1$

8. $y = x$

9. $y = x^2$

10. $y = \frac{1}{6}x^3$

11. $y = \frac{1}{x^2}$

12. $y = \sin x$

13. $y = \cos x$

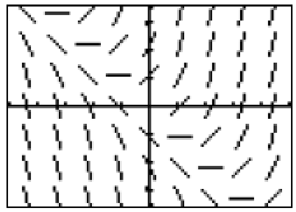
14. $y = \ln|x|$

9.

The calculator drawn slope field for the differential equation $\dfrac{dy}{dx} = x + y$ is shown in the figure below.

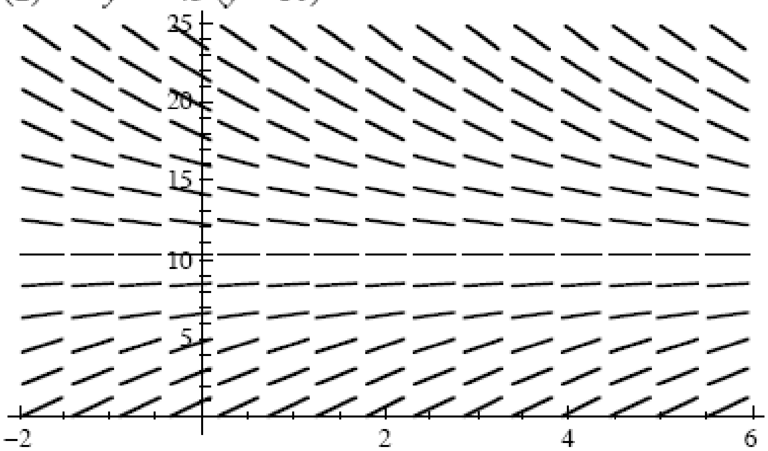
(a) Sketch the solution curve through the point (0, 1).

(b) Sketch the solution curve through the point $(-3, 0)$.



Draw possible solution curves to the differential equations below going through each of the given points. Be sure to label each one.

(2) $y' = -.3\,(y - 10)$

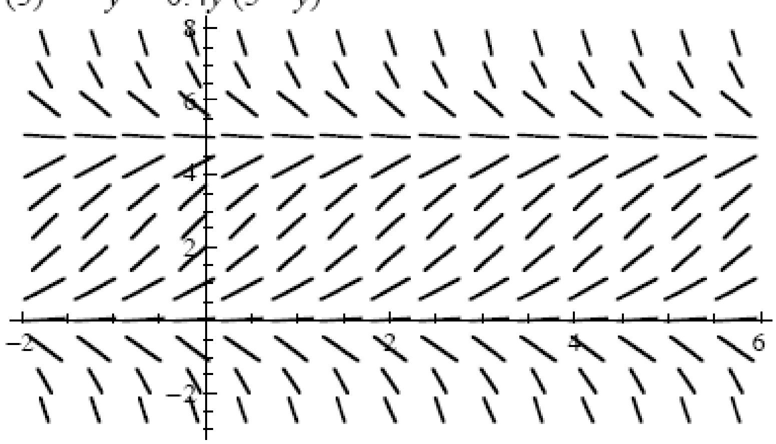


$A(0, 25)$

$B(0, 10)$

$C(0, 0)$

(3) $y' = 0.4y\,(5 - y)$

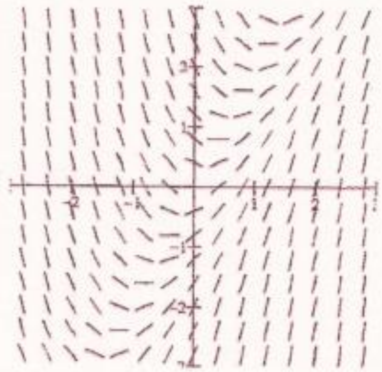


$A(0, 8)$

$B(0, 1)$

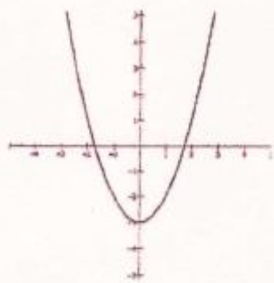
$C(0, -1/2)$

10. Slope field $y' = 2x - y$

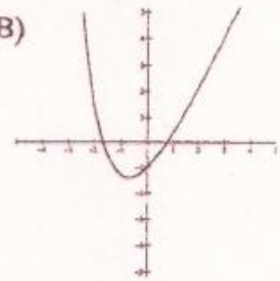


Which graph could be a solution of the differential equation that is shown above?

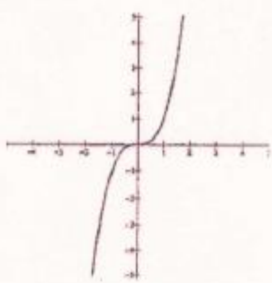
(A)



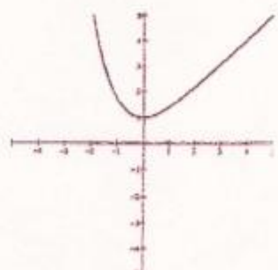
(B)



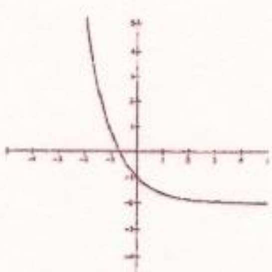
(C)



(D)



(E)



11.

A.______ $\frac{dy}{dx} = y^2$ B.______ $\frac{dy}{dx} = 1$

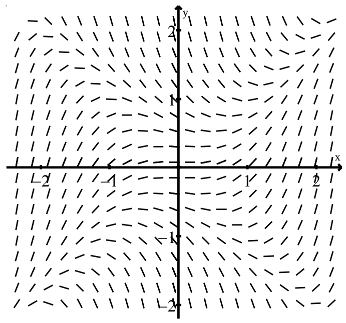
C.______ $\frac{dy}{dx} = x^2$ D.______ $\frac{dy}{dx} = x + y$

E.______ $\frac{dy}{dx} = \sin(x)$ F.______ $\frac{dy}{dx} = -y$

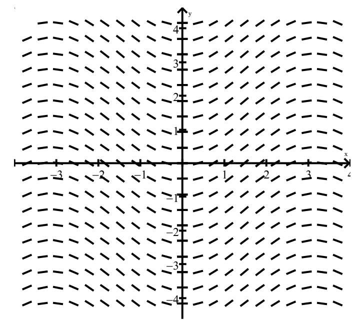
G.______ $\frac{dy}{dx} = 1 - x$ H.______ $\frac{dy}{dx} = 1 - y^2$

I. ______ $\frac{dy}{dx} = 1 + y$ J.______ $\frac{dy}{dx} = x^2 - y^2$

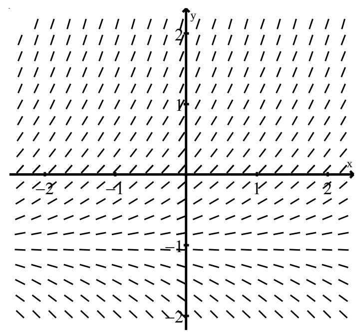
I



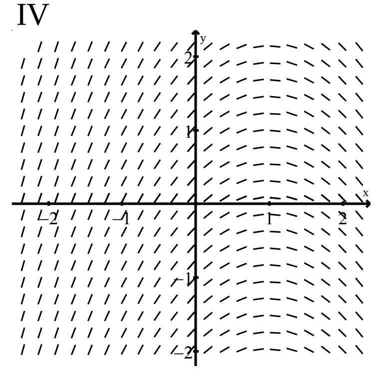
II



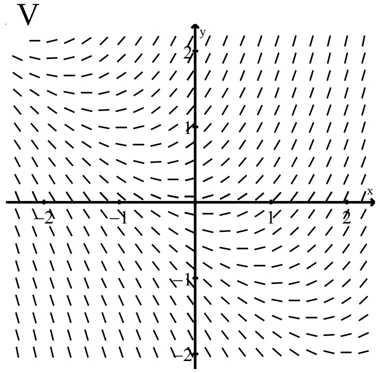
III



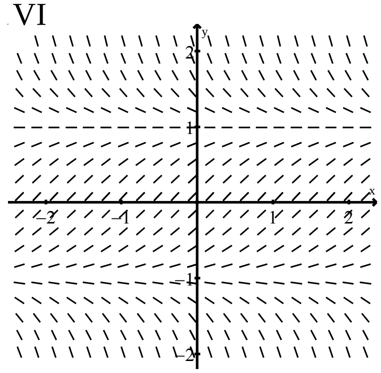
IV



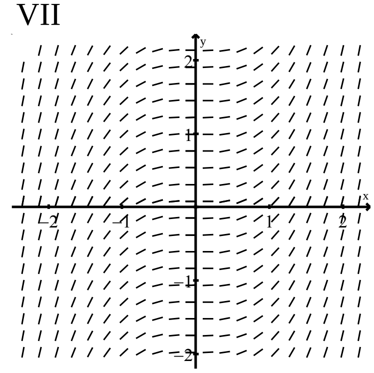
V



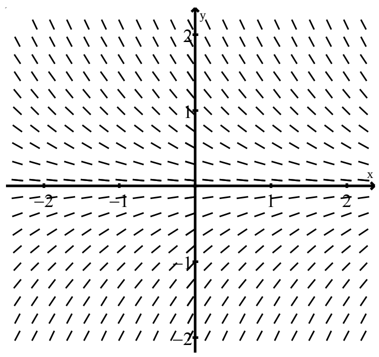
VI



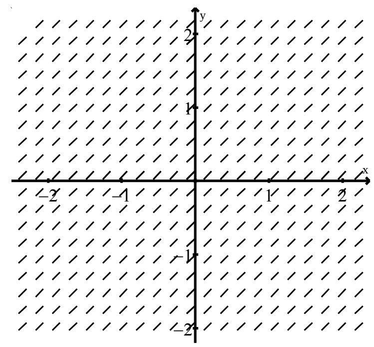
VII



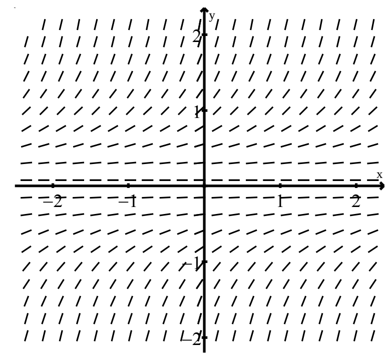
VIII



IX



X



12. The slope field for the differential equation $\frac{dy}{dx} = \frac{x^2y + y^2x}{3x+y}$ will have horizontal segments when

(A) $x = 0$ or $y = 0$, only

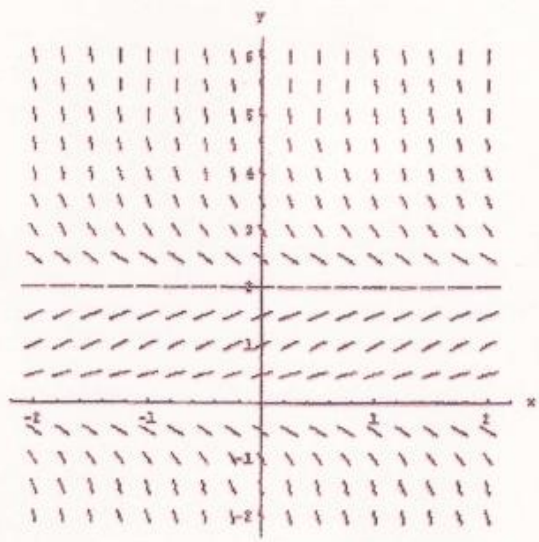
(B) $y = -x$, only

(C) $y = -3x$, only

(D) $y = 5$, only

(E) $x = 0$, or $y = 0$, or $y = -x$

13.



The slope field for a differential equation $\frac{dy}{dx} = f(y)$ is shown in the figure above. Which statement is true about $y(x)$?

I. If $y(0) > 2$ then $\lim\limits_{x \to \infty} y(x) \approx 2$

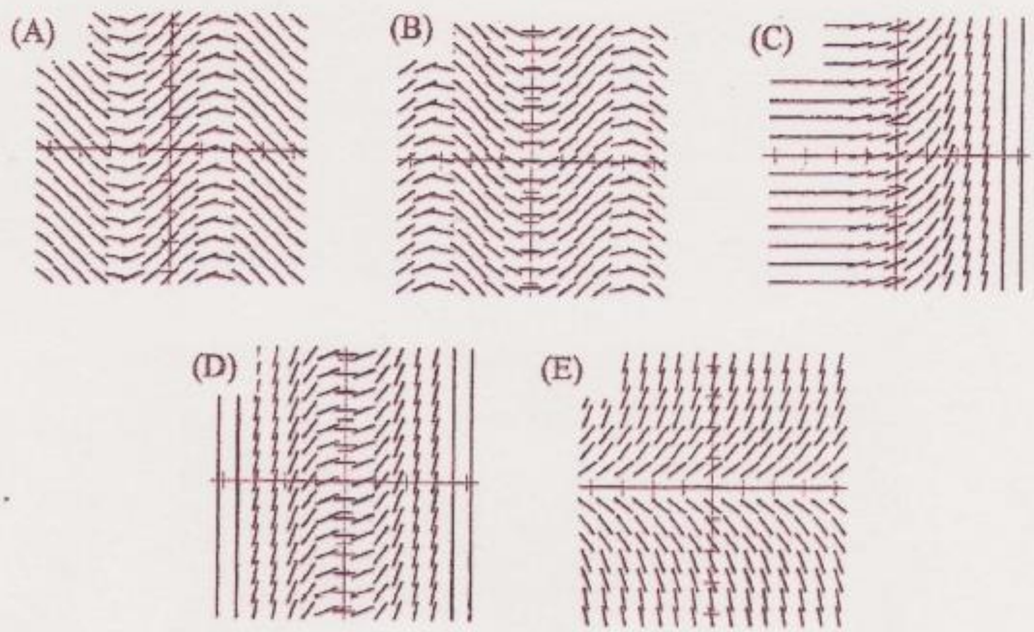
II. If $0 < y(0) < 2$ then $\lim\limits_{x \to \infty} y(x) \approx 2$

III. If $y(0) < 2$ then $\lim\limits_{x \to \infty} y(x) \approx 2$

(A) I only (B) II only (C) III only (D) I and II only (E) I, II and III

14. Which choice represents the slope field for $dy/dx = \sin x$?

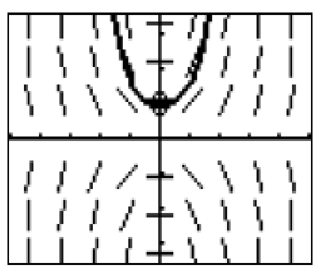
Note: All graphs are for $-4 < x < 4$ and $-4 < y < 4$



15. The calculator drawn slope field for the differential equation $\frac{dy}{dx} = xy$ is shown in the figure below. The solution curve passing through the point (0, 1) is also shown.

(a) Sketch the solution curve through the point (0, 2).

(b) Sketch the solution curve through the point $(0, -1)$.



Exercises and Problems for Section 11.2

Exercises

1. Sketch three solution curves for each of the slope fields in Figures 11.14 and 11.15.

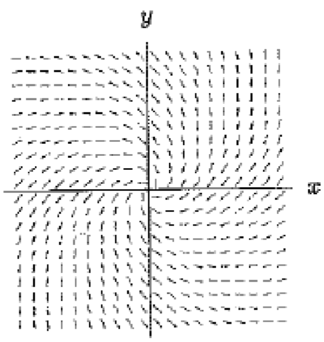


Figure 11.14

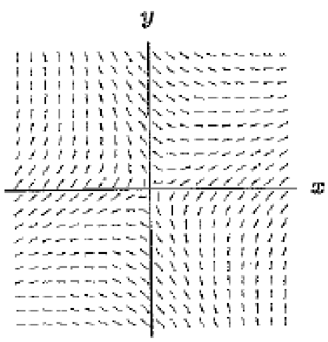


Figure 11.15

2. The slope field for the equation $y' = x(y-1)$ is shown in Figure 11.16.

(a) Sketch the solutions passing through the points
(i) $(0,1)$ (ii) $(0,-1)$ (iii) $(0,0)$

(b) From your sketch, write down the equation of the solution with $y(0) = 1$.

(c) Check your solution to part (b) by substituting it into the differential equation.

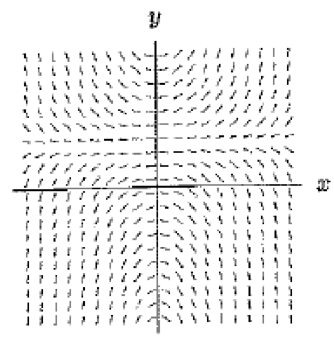


Figure 11.16

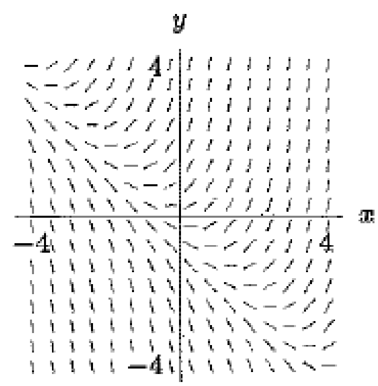
3. The slope field for the equation $y' = x + y$ is shown in Figure 11.17.

(a) Carefully sketch the solutions that pass through the points

(i) $(0,0)$ (ii) $(-3,1)$ (iii) $(-1,0)$

(b) From your sketch, write the equation of the solution passing through $(-1,0)$.

(c) Check your solution to part (b) by substituting it into the differential equation.



4. The slope field for the equation $dP/dt = 0.1P(10-P)$, for $P \geq 0$, is in Figure 11.18.

   (a) Plot the solutions through the following points:

   (i) $(0,0)$ (ii) $(1,4)$ (iii) $(4,1)$
   (iv) $(-5,1)$ (v) $(-2,12)$ (vi) $(-2,10)$

   (b) For which positive values of P are the solutions increasing? Decreasing? What is the limiting value of P as t gets large?

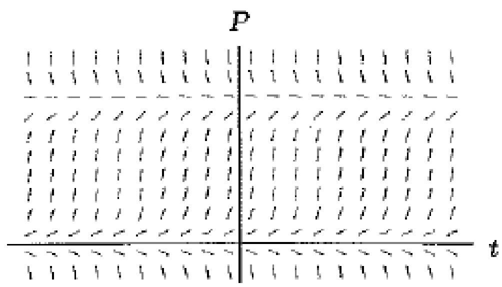


Figure 11.18

5. Figure 11.19 shows the slope field for the equation $y' = (\sin x)(\sin y)$.

   (a) Sketch the solutions that pass through the points: (i) $(0,-2)$ (ii) $(0,\pi)$.

   (b) What is the equation of the solution that passes through $(0, n\pi)$, where n is any integer?

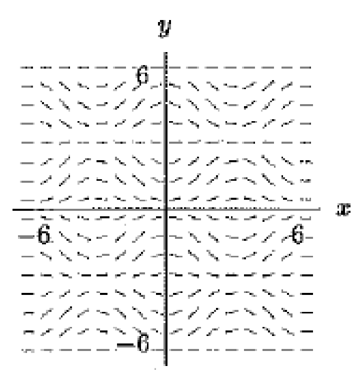


Figure 11.19: $y' = (\sin x)(\sin y)$

6. One of the slope fields in Figure 11.20 has the equation $y' = (x+y)/(x-y)$. Which one?

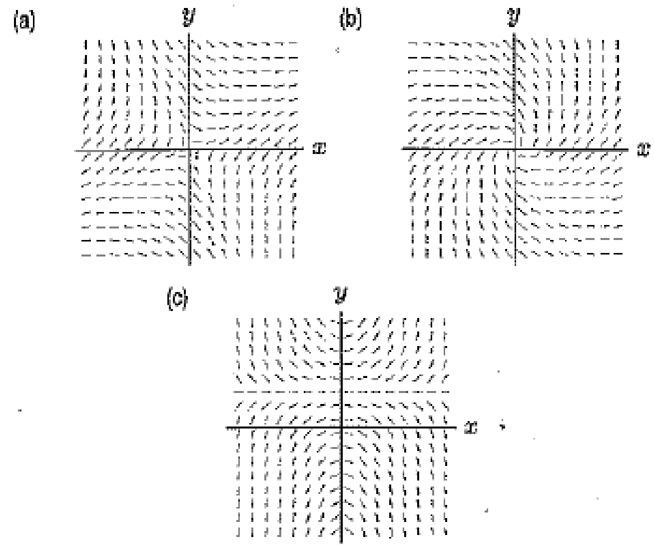


Figure 11.20

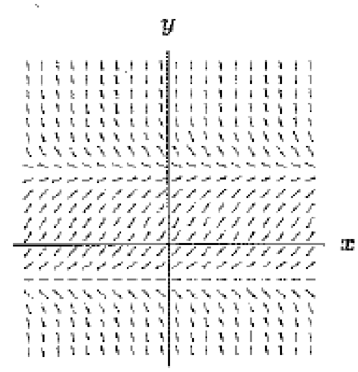
7. The slope field for $y' = 0.5(1+y)(2-y)$ is shown in Figure 11.21.

   (a) Plot the following points on the slope field:

   (i) the origin (ii) $(0,1)$ (iii) $(1,0)$
   (iv) $(0,-1)$ (v) $(0,-5/2)$ (vi) $(0,5/2)$

   (b) Plot solution curves through the points in part (a).

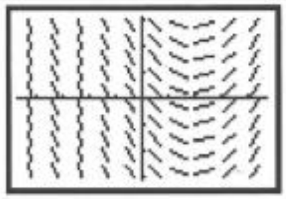
   (c) For which regions are all solution curves increasing? For which regions are all solution curves decreasing? When can the solution curves have horizontal tangents? Explain why, using both the slope field and the differential equation.



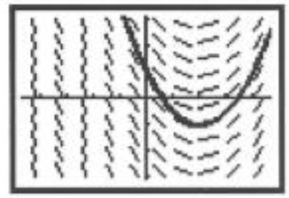
Hint: The answer to 3b on page 20 should be the equation of a line.

WHERE DOES A MATHEMATICIAN PICK HIS DERIVATIVES?

Given a function f(x) such that $f'(x) = x - 2$, An equation for f(x) is $f(x) = \frac{1}{2}x^2 - 2x + c$ where c is any constant.



a sketch of line segments tangent to f(x) for different values of c.



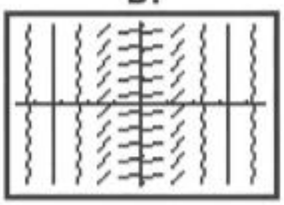
$f(x) = \frac{1}{2}x^2 - 2x + 1$ through the point (0, 1).

Match each derivative f'(x) with a graph of line segments tangent to possible functions f(x).

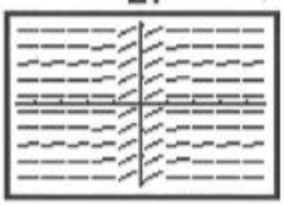
1) $f'(x) = 2x + 2$	2) $f'(x) = x^2$	3) $f'(x) = y$	4) $f'(x) = x + y$
5) $f'(x) = \frac{x}{y}$	6) $f'(x) = -\frac{x}{y}$	7) $f'(x) = e^{-x^2}$	8) $f'(x) = -\sin(x)$

Slope fields.

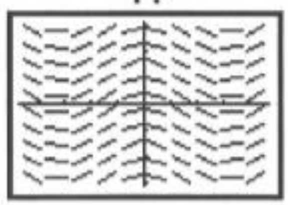
D.



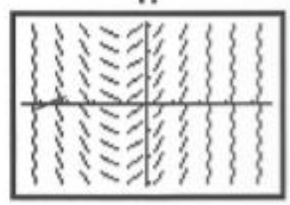
E.



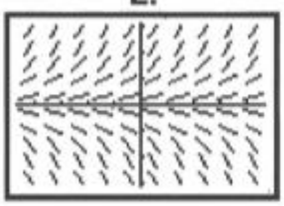
F.



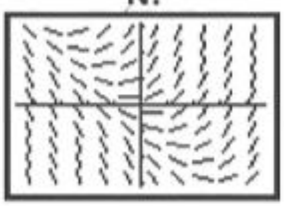
I.



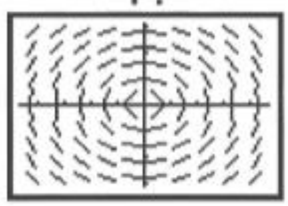
L.



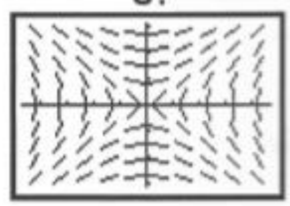
N.



P.



S.



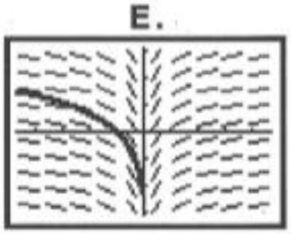
9) Which of the functions f(x) below would satisfy $f'(x) = x^2$.			
A. $f(x) = \frac{1}{2}x^2 + c$	G. $f(x) = x^3 + c$	O. $f(x) = \frac{1}{3}x^3 + c$	T. $f(x) = 3x^3 + c$

Match each derivative f'(x) with a graph of the function f(x) that passes through the given point.

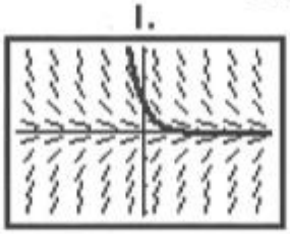
10) $f'(x) = \frac{1}{x}$; (e, 1)	11) $f'(x) = \frac{1}{x}$; (–e, 1)	12) $f'(x) = -2y$; (0, –1)	13) $f'(x) = -2y$ (0, 1)

Functions f(x).

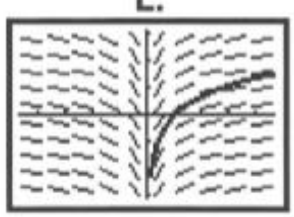
E.



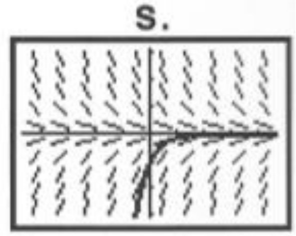
I.



L.



S.



1	4

12	10	9	6	7

8	13	11	3	2	5

11.4 Differential EQ: Separate, Integrate & Solve

Steps in Solving Differential Equations

1. Separate the variables.
2. Integrate both sides of the equation. Be sure to include a constant integration on one side. Traditionally this is included on the side of the independent variable since you will be solving for the dependent variable.
3. If an initial condition is included, substitute and solve for *c*.
4. Solve for the dependent variable.

Example 1: Solve the differential equation $\frac{dy}{dx} = \frac{2x}{y}$ (Fill in the reasons for each step)

SOLUTION STEPS:	REASON
$\frac{dy}{dx} = \frac{2x}{y}$	**Given**
$ydy = 2xdx$	
$\int y\,dy = \int 2x\,dx$	
$\frac{1}{2}y^2 + C_2 = x^2 + C_1$	
$\frac{1}{2}y^2 = x^2 + C$	
$y^2 = 2x^2 + C$	
$y = \pm\sqrt{2x^2 + C}$	

This is a general solution to the differential equation because C is part of the solution.

Example 2: Solve the differential equation $\frac{dy}{dx} = xy$ that satisfies the initial condition $y(0) = 2$.
(Fill in the reasons for each step)

SOLUTION STEPS:	REASON
$\frac{dy}{dx} = xy$	**Given**
$\frac{dy}{y} = xdx$	
$\int \frac{dy}{y} = \int xdx$	
$\ln\lvert y\rvert = \frac{1}{2}x^2 + c$	
$\ln 2 = \frac{1}{2}(0)^2 + c$	
$\ln 2 = c$	
$\ln\lvert y\rvert = \frac{1}{2}x^2 + \ln 2$	
$e^{\ln\lvert y\rvert} = e^{\left(\frac{1}{2}x^2 + \ln 2\right)}$	
$\lvert y\rvert = e^{\left(\frac{1}{2}x^2\right)} \cdot e^{\ln 2}$	
$y = \pm 2e^{\frac{1}{2}x^2}$	
$y = 2e^{\frac{1}{2}x^2}$	

This is a particular solution to the differential equation.

**** Make note of why the solution is the positive equation rather than the negative equation. ****

What would the initial condition have needed to be in order for the solution to be the negative equation?

Example 3: Solve the equation $\frac{dy}{dx} = (x-2)y^2$

Example 4: Solve the differential equation $\frac{dy}{dx} = \frac{6x^2}{2y + \cos y}, y(1) = \pi$

Example 5: Solve the differential equation $y' = \frac{1+x}{xy}$ if $x > 0,\ y(1) = -4$

6. AB73-37.

If $\dfrac{dy}{dx} = 4y$ and if $y = 4$ when $x = 0$, then $y =$

(A) $4e^{4x}$ (B) e^{4x} (C) $3 + e^{4x}$ (D) $4 + e^{4x}$ (E) $2x^2 + 4$

7. AB93-33

If $\dfrac{dy}{dx} = 2y^2$ and if $y = -1$ when $x = 1$, then when $x = 2, y =$

(A) $-\dfrac{2}{3}$ (B) $-\dfrac{1}{3}$ (C) 0 (D) $\dfrac{1}{3}$ (E) $\dfrac{2}{3}$

8. AB93-42

A puppy weights 2.0 pounds at birth and 3.5 pounds two months later. If the weight of the puppy during its first 6 months is increasing at a rate proportional to its weight, then how much will the puppy weigh when it is 3 months old?

(A) 4.2 lbs. (B) 4.6 lbs. (C) 4.8 lbs (D) 5.6 lbs (E) 6.5 lbs

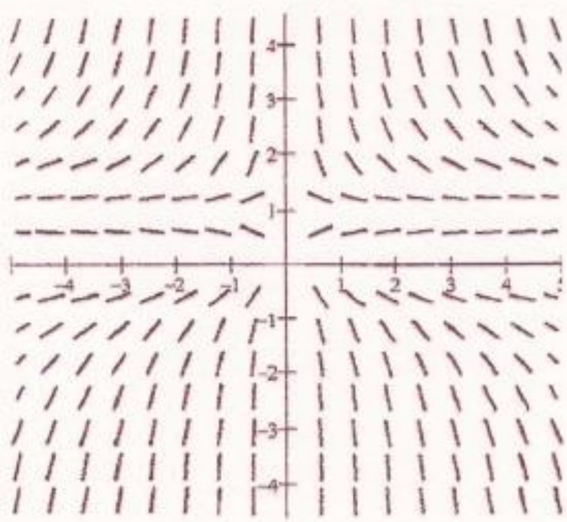
9. Consider the differential equation $\frac{dy}{dx} = \frac{y - y^2}{x}$ for all $x \neq 0$.

(a) Verify that $y = \frac{x}{x+C}$, $x \neq -C$ is a general solution for the given differential equation and show that all solutions contain (0,0).

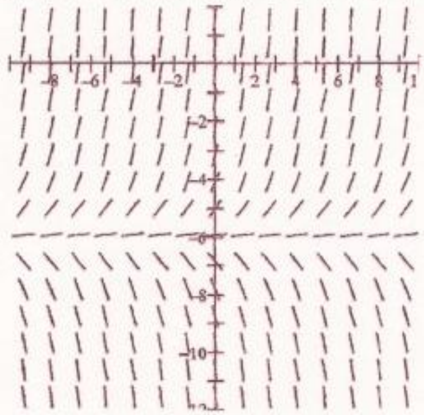
(b) Write an equation of the particular solution that contains the point (1, 2). and find the value of $\frac{dy}{dx}$ at (0,0) for this solution.

(c) Write an equation of the vertical and horizontal asymptotes of the particular solution found in (b).

(d) The slope field for the given differential equation is provided. Sketch both branches of the particular solution curve that passes through the point (1,2).



10. Consider the differential equation $dy/dx = 6 + y$ defined for all real numbers. The slope field for this equation is shown below in the window $-10 < x < 10$ by $-12 < y < 2$.



(a) Find the general solution of the differential equation in terms of an arbitrary constant C.

(b) Find the particular solution of the differential equation that meets the initial condition that $y = 0$ when $x = 0$. Sketch this solution on the slope field.

(c) Find the particular solution of the differential equation that meets the initial condition that $y = -8$ when $x = 0$. Sketch this solution on the slope field.

(d) The slope field indicates that for some of the solutions $\lim_{x\to\infty} y = \infty$ and for other solutions $\lim_{x\to\infty} y = -\infty$. Determine the values of C for which $\lim_{x\to\infty} y = \infty$. Show your reasoning.

Additional Practice

Solve the differential equations subject to the prescribed initial condition.

1. $\frac{dy}{dx} = \frac{x^2+1}{x^2};\ x=1, y=1$

2. $\frac{dy}{dx} = 2xy^2;\ x=1, y=1$

3. $\frac{dy}{dx} = x\sqrt{y};\ x=0, y=1$

4. $\frac{dy}{dx} = \frac{1}{x^2\sqrt{y+1}};\ x=1, y=-1$

5. $\frac{dy}{dx} = 2y-4;\ y(2)=5$

6. $\frac{dy}{dx} = xy^2\cos(x^2);\ y(0)=1$

7. The graph of $y = f(x)$ passes through the point (9, 4). Also, the line tangent to the graph at any point (x, y) has the slope $3\sqrt{x}$. Find f.

11.5 Growth and Decay

A reasonable differential equation expressing the change in population over time is

$$\frac{dP}{dt} = kP$$

The constant k is the exponential growth or decay constant. If $k > 0$, the population in growing. If $k < 0$, the population is shrinking. If we solve this differential equation for P, we get

$$\frac{dP}{P} = kdt$$
$$\int \frac{dP}{P} = \int kdt$$
$$\ln|P| = kt + c$$
$$|P| = e^{kt+c}$$
$$P = Ce^{kt}$$

If at $t = 0$, the population is P_0, we get $P_0 = Ce^{k(0)}$, so $C = P_0$ and $P = P_0 e^{kt}$. (Remember PERT from Precalculus?)

<u>Example 1:</u> The number of bacteria in a rapidly growing culture was estimated to be 10,000 at noon and 40000 after 2 hours. How many bacteria will there be at 5 pm?

<u>Example 2:</u> (Decay) Carbon 14 is radioactive and decays at a rate proportional to the amount present. Its half-life is 5730 years. If 10 grams were present originally, how much will be left after 2000 years?

Example 3: A colony of bacteria is grown under ideal conditions in a laboratory so that the population increases exponentially with time. At the end of 3 hours there are 10,000 bacteria. At the end of 5 hours there are 40,000 bacteria. How many bacteria were present initially?

Example 4: Pharmacologists have shown that penicillin leaves a person's bloodstream at a rate proportional to the amount present.

a. Express this statement as a differential equation.

b. Find the decay constant if 50 mg of penicillin remains in the bloodstream 7 hours after an initial injection of 450 mg.

c. Under the hypothesis of (b), at what time was 200 mg of penicillin present?

Newton's Law of Cooling

There is a differential equation that is closely related to the growth/decay equation. It is $\frac{dy}{dt} = k(y-b)$ where k and b are constants and $k \neq 0$. This differential equation describes a quantity y whose rate of change is proportional to the difference $(y - b)$. Solve this equation:

The general solution to this differential equation is $y(t) = b + Ce^{kt}$.

The most common application that utilizes this differential equation is Newton's Law of Cooling. Let $y(t)$ be the temperature of a hot object that is cooling off (or a cold object that is warming up) in an environment where the ambient temperature is T_0. Newton assumed that the rate of cooling is proportional to the temperature difference $y - T_0$.

Example 1: Joe's automobile engine runs at $100^\circ C$. On a day when the outside temperature is $21^\circ C$, he turns off the ignition and notes that five minutes later, the engine has cooled to $70^\circ C$.

a. Determine the engine's cooling constant *k*.

b. What is the formula for $y(t)$?

c. When will the engine cool to $40^\circ C$?

Example 2: On Tuesday night, Ms. Ritz accidently left her computer in her car overnight. She was so worried the next morning when she went outside and saw frost on her car she knew that her poor computer had froze out there all night! Who would have ever thought; it had been 70 degrees the week before! Ms. Ritz brought the computer inside the house which stays a constant 65° F room temperature. After 10 minutes, the computer warmed to 35° F. After another 10 minutes, the temperature of the computer was 50° F. Use Newton's Law of Heating and Cooling to estimate the following:

a. What was the computer's initial temperature?

b. When was the computer 63° F?

c. What was the temperature of the computer after 1 hour?