CH 7.1 Review Integration by U-Substitution

① POWER RULE

A) $\int x\,dx = \frac{1}{2}x^2 + c$

B) $\int 5\,dx = 5x + c$

C) $\int x^5\,dx = \frac{1}{6}x^6 + c$

D) $\int \sqrt{x}\,dx = \frac{2}{3}x^{3/2} + c = \frac{2}{3}\sqrt{x^3} + c$

E) $\int 5x^{-2/3}\,dx = 15x^{1/3} + c = 15\sqrt[3]{x} + c$

F) $\int x^{-4}\,dx = -\frac{1}{3}x^{-3} + c = \frac{-1}{3x^3} + c$

G) $\int 3x^{-2}\,dx = -1 \cdot 3x^{-1} + c = \frac{-3}{x} + c$

H) $\int 3x^{0.7}\,dx = \frac{3x^{1.7}}{1.7} + c = \frac{30x^{1.7}}{17} + c$

I) $\int \sqrt[3]{x}\,dx = \int x^{1/3}\,dx = \frac{3}{4}x^{4/3} + c$

J) $\int x^e\,dx = \frac{x^{e+1}}{e+1} + c$

Thinking BACKWARDS

K) $\int \sin x\,dx = -\cos x + c$

L) $\int \sec^2 x\,dx = \tan x + c$

M) $\int dx = x + c$

N) $\int \csc^2 x\,dx = -\cot x + c$

O) $\int \cos x\,dx = \sin x + c$

P) $\int \sec x \tan x\,dx = \sec x + c$

Q) $\int a^x\,dx = \frac{a^x}{\ln a} + c$

R) $\int e^x\,dx = e^x + c$

S) $\int 5^x\,dx = \frac{5^x}{\ln 5} + c$

T) $\int \csc x \cot x\,dx = -\csc x + c$

② SUBSTITUTION RULE:

A) $\int \sin(3x)\,dx = \frac{1}{3}\int \sin(u)\,du$

$u = 3x$

$du = 3dx$

$\frac{1}{3}du = dx$

$= -\frac{1}{3}\cos u + c$

$= -\frac{1}{3}\cos(3x) + c$

B) $\int (2x-1)e^{(x^2-x)}\,dx = \int e^u\,du$

$u = x^2 - x$

$du = (2x-1)dx$

$= e^u + c$

$= e^{x^2-x} + c$

C) $\int \sec^2(7x)\,dx = \frac{1}{7}\int \sec^2(u)\,du$

$u = 7x$

$du = 7dx$

$\frac{1}{7}du = dx$

$= \frac{1}{7}\tan(u) + c$

$= \frac{1}{7}\tan(7x) + c$

D) $\int \frac{\cos\sqrt{x}}{\sqrt{x}}\,dx = \int \cos(u)\,du$

$u = \sqrt{x}$

$du = \frac{1}{2\sqrt{x}}dx$

$2du = \frac{dx}{\sqrt{x}}$

$= \sin(u) + c$

$= \sin(\sqrt{x}) + c$

$\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx = -\int \frac{du}{u} =$

$u = \cos x$

$du = -\sin x\,dx$

$-du = \sin x\,dx$

$= -\ln|u| + c$

$= -\ln|\sin x| + c$

$= \ln|\csc x| + c$

E) $\int \sin^3(x)\cos(x)\,dx =$

$u = \sin x$

$du = \cos x\,dx$

$\int u^3\,du$

$= \frac{1}{4}u^4 + c$

$= \frac{1}{4}\sin^4 x + c$

*F) $\int \tan(3x)\,dx$

$u = 3x$

$du = 3dx$

$\frac{1}{3}du = dx$

$= \frac{1}{3}\int \tan(u)\,du$

$= \frac{1}{3}\ln|\csc u| + c$

$= \frac{1}{3}\ln|\csc(3x)| + c$

G) $\int \frac{\ln x}{x}\,dx = \int u\,du$

$u = \ln x$

$du = \frac{1}{x}dx$

$= \frac{u^2}{2} + c$

$= \frac{1}{2}(\ln|x|)^2 + c$

H) $\int \frac{3x-9}{x^2-6x+7}\,dx = \int \frac{3(x-3)\,dx}{x^2-6x+7}$

$u = x^2 - 6x + 7$

$du = 2x - 6$

$\frac{1}{2}du = (x-3)dx$

$= \frac{3}{2}\int \frac{1}{u}\,du$

$= \frac{3}{2}\ln|u| + c$

$= \frac{3}{2}\ln|x^2-6x+7| + c$

CH 7.1 Review (continued)

(2) J) $\int \frac{1}{1+(2x)^2}\,dx = \frac{1}{2}\int \frac{1}{1+u^2}\,du$

$u = 2x$ $\quad = \frac{1}{2}\arctan(u) + c$

$du = 2dx$

$\frac{1}{2}du = dx$ $\quad = \frac{1}{2}\arctan(2x) + c$

K) $\int \frac{1}{\sqrt{1-9x^2}}\,dx = \int \frac{1}{\sqrt{1-(3x)^2}}\,dx$

$u = 3x$ $\quad = \frac{1}{3}\int \frac{1}{\sqrt{1-u^2}}\,du$

$du = 3dx$

$\frac{1}{3}du = dx$ $\quad = \frac{1}{3}\arcsin(u) + c$

$= \frac{1}{3}\arcsin(3x) + c$

L) $\int x\sin^3(x^2)\cos(x^2)\,dx$

$u = \sin x^2$

$du = \cos(x^2)(2x)\,dx$

$\frac{1}{2}du = x\cos(x^2)\,dx$

$= \frac{1}{2}\int u^3\,du = \frac{1}{8}u^4 + c$

$= \frac{1}{8}\sin^4(x^2) + c$

M) $\int \frac{(x-3)\,dx}{\sqrt{x^2-6x+7}} = \frac{1}{2}\int \frac{du}{\sqrt{u}}$

$u = x^2-6x+7$ $\quad = \frac{1}{2}\int u^{-\frac{1}{2}}\,du$

$du = 2x-6\,dx$ $\quad = u^{\frac{1}{2}} + c$

$\frac{1}{2}du = x-3\,dx$

$= \sqrt{x^2-6x+7} + c$

(3) DEFINITE INTEGRALS w/ usub:

A) $\int_0^5 x\sqrt{x^2+4}\,dx = \frac{1}{2}\int_4^{29}\sqrt{u}\,du = \cancel{\frac{1}{2}}\frac{\cancel{2}}{3}u^{3/2}\Big|_4^{29} = \boxed{\frac{1}{3}\left(\sqrt{29^3}-\sqrt{4^3}\right)}$ exact

$u = x^2+4 \longrightarrow x=0 \therefore u=4$

$du = 2x\,dx$ $\quad x=5 \therefore u=29$

$\frac{1}{2}du = x\,dx$

Approx ≈ 49.389

≈ 49.390

B) $\int_0^{\pi} \sin\theta(\cos\theta+5)^7\,d\theta = -\int_6^4 u^7\,du = +\int_4^6 u^7\,du = \frac{1}{8}u^8\Big|_4^6$

$u = \cos\theta+5 \longrightarrow \theta=0 \therefore u=6$

$du = -\sin\theta\,d\theta$ $\quad \theta=\pi \therefore u=4$

$-du = \sin\theta\,d\theta$

$= \boxed{\frac{1}{8}(6^8-4^8)}$

$= \frac{1}{8}(2^8\cdot 3^8 - 2^8\cdot 2^8)$

$= \frac{2^8}{2^3}(3^8-2^8)$

$= 2^5(3^8-2^8) = \boxed{32(3^8-2^8)}$ exact

$= 201{,}760$

C) $\int_0^1 \frac{x}{5x^2+1}\,dx = \frac{1}{10}\int_1^6 \frac{1}{u}\,du = \frac{1}{10}\ln|u|\Big|_1^6$

$u = 5x^2+1 \longrightarrow x=0 \therefore u=1$ $\quad = \frac{1}{10}\left(\ln(6)-\ln(1)\right)$

$du = 10x\,dx$ $\quad x=1 \therefore u=6$

$\frac{1}{10}du = x\,dx$ $\quad = \frac{1}{10}(\ln 6 - 0)$

exact $= \boxed{\frac{1}{10}\ln 6}$

APPROX ≈ 0.179

CH 7.1 Review (continued)

(3) D) $\int_0^2 \left(2x^3 - 6x + \frac{3}{x^2+1}\right)dy$

$= \frac{x^4}{2} - 3x^2 + 3\arctan(x)\Big|_0^2$

no u-sub necessary

$= [8 - 12 + 3\arctan(2)] - [0 + 3\arctan(0)]$

$= -4 + 3\arctan(2) - (0)$

$= \boxed{-4 + 3\arctan(2)}$

E) $\int_1^9 \frac{2t^2 + t^2\sqrt{t} - 1}{t^2}\,dt$

$\int_1^9 (2 + \sqrt{t} - t^{-2})\,dt$

$2t + \frac{2}{3}t^{3/2} + \frac{1}{t}\Big|_1^9$

$\left(18 + 18 + \frac{1}{9}\right) - \left(2 + \frac{2}{3} + 1\right)$

$\left(36 + \frac{1}{9}\right) - \left(3 + \frac{2}{3}\right)$ $\frac{4}{9}$

$33 - \frac{1}{3} = \boxed{32\frac{2}{3}}$

(4) ALGEBRA:

(A) $\int x^2(x^2-3)^2\,dx$

$= \int x^2(x^4 - 6x^2 + 9)\,dx$

$= \int x^6 - 6x^4 + 9x^2\,dx$

$= \frac{1}{7}x^7 - \frac{6}{5}x^5 + 3x^3 + C$

(B) $\int \frac{(x+1)^2}{x}\,dx$

$= \int \left(\frac{x^2+2x+1}{x}\right)dx$

$= \int x + 2 + \frac{1}{x}\,dx$

$= \frac{1}{2}x^2 + 2x + \ln|x| + C$

(C) $\int \frac{3x-5}{\sqrt{x}}\,dx$

$\int 3\sqrt{x} - 5x^{-1/2}\,dx$

$= 2x^{3/2} - 10\sqrt{x} + C$

$\boxed{\text{MC}}$ $\boxed{\text{NONCALCULATOR}}$

(7) $y = x^2\sqrt{x^3+1}$ on $[0,2]$ Avg Value $= \frac{1}{2}\int_0^2 x^2\sqrt{x^3+1}\,dx$

$u = x^3+1$

$du = 3x^2\,dx$

$\frac{1}{3}du = x^2\,dx$

$= \frac{1}{6}\int_1^9 \sqrt{u}\,du = \frac{1}{\cancel{6}_3}\cdot\frac{\cancel{2}}{3}u^{3/2}\Big|_1^9$

$= \frac{1}{9}\left(u^{3/2}\right)\Big|_1^9 = \frac{1}{9}\left(9^{3/2} - 1^{3/2}\right)$

$= \frac{1}{9}(27 - 1) = \boxed{\frac{26}{9}}$ (A)

(10) $\int \frac{x\,dx}{\sqrt{3x^2+5}} = \frac{1}{6}\int \frac{1}{\sqrt{u}}\,du$

$u = 3x^2+5$

$du = 6x\,dx$

$\frac{1}{6}du = x\,dx$

$= \frac{1}{6}\int u^{-1/2}\,du$

$= \frac{1}{6}\cdot\frac{2}{1}u^{1/2} + C$

$= \frac{1}{3}\sqrt{3x^2+5} + C$

(D)

(12) $\int_0^{\pi/2} \frac{\cos\theta}{\sqrt{1+\sin\theta}}\,d\theta$ $\int_1^2 \frac{1}{\sqrt{u}}\,du = \int_1^2 u^{-1/2}\,du$

$u = 1+\sin\theta$

$du = \cos\theta\,d\theta$

$x=0 \rightarrow u=1$

$x=\frac{\pi}{2} \rightarrow u=2$

$2\cdot u^{1/2}\Big|_1^2$

$= 2(\sqrt{2} - \sqrt{1})$

$= 2(\sqrt{2} - 1)$

(D)

CH 7 Review (continued)

NO CALCULATOR MC.

(14) $\int_2^3 \frac{x}{x^2+1}dx = \frac{1}{2}\int_5^{10} \frac{1}{u}du = \frac{1}{2}\ln|u|\Big|_5^{10}$

$u = x^2+1$

$du = 2x\,dx$

$\frac{1}{2}du = x\,dx$

$= \frac{1}{2}(\ln|10| - \ln|5|)$

$= \frac{1}{2}\ln(2)$

(B)

* (19) $\int_0^{\sqrt{3}} \frac{dx}{\sqrt{4-x^2}} = \int_0^{\sqrt{3}} \frac{dx}{2\sqrt{1-\left(\frac{x}{2}\right)^2}} = 1\int_0^{\frac{\sqrt{3}}{2}} \frac{du}{\sqrt{1-u^2}}$

$u = \frac{x}{2}$

$du = \frac{1}{2}dx$

$2du = dx$

$= \arcsin(x)\Big|_0^{\sqrt{3}/2}$

$= \arcsin\left(\frac{\sqrt{3}}{2}\right) - \arcsin(0)$

$= \frac{\pi}{3} - 0 = \frac{\pi}{3}$ (A)

(22) $\int_0^{\pi/4} \frac{e^{\tan x}}{\cos^2 x}dx = \int_0^{\pi/4} e^{\tan x}\sec^2 x\,dx$

$u = \tan x$

$du = \sec^2 x\,dx$

$\int_0^1 e^u du = e^u\Big|_0^1 = e^1 - e^0 = e - 1$ (C)

AP FRQ #1 (1991)

a) $f'(x) = 0$

$f''(x) = 24x - 18$

$f'(x) = \int(24x-18)dx$

$= 12x^2 - 18x + c$

$f'(1) = -6 \therefore -6 = 12 - 18 + c$

$c = 0$

$f'(x) = 12x^2 - 18x + 0 = 0$

$6x(2x-3) = 0$

$\boxed{x = 0 \quad x = \frac{3}{2}}$

b) $f(2) = 0 \quad f(x) = ?$

$f(x) = \int 12x^2 - 18x\,dx$

$f(x) = 4x^3 - 9x^2 + c$

$x^2(4x-9) + c\Big|_{x=2} = 0$

$4(8-9) + c = 0$

$-4 + c = 0$

$c = 4$

$\boxed{f(x) = 4x^3 - 9x^2 + 4}$

c) Avg Value of f on [1, 3]

$\frac{1}{3-1}\int_1^3 4x^3 - 9x^2 + 4\,dx$

$\frac{1}{2}\left(x^4 - 3x^3 + 4x\right)\Big|_1^3$

$\frac{1}{2}\left(x\left(x^3 - 3x^2 + 4\right)\right)\Big|_1^3$

$\frac{1}{2}\left[3(4) - 1(2)\right]$

$\frac{1}{2}(12-2) = \boxed{5}$

2004 SCORING GUIDELINES **Question 1**

Traffic flow is defined as the rate at which cars pass through an intersection, measured in cars per minute. The traffic flow at a particular intersection is modeled by the function F defined by

$$F(t) = 82 + 4\sin\left(\frac{t}{2}\right) \text{ for } 0 \le t \le 30,$$

where $F(t)$ is measured in cars per minute and t is measured in minutes.

(a) $\int_0^{30} F(t)\,dt = 2474$ cars	$3: \begin{cases} 1: \text{limits} \\ 1: \text{integrand} \\ 1: \text{answer} \end{cases}$
(b) $F'(7) = -1.872$ or -1.873 Since $F'(7) < 0$, the traffic flow is decreasing at $t = 7$.	1 : answer with reason
(c) $\frac{1}{5}\int_{10}^{15} F(t)\,dt = 81.899$ cars/min	$3: \begin{cases} 1: \text{limits} \\ 1: \text{integrand} \\ 1: \text{answer} \end{cases}$
(d) $\frac{F(15) - F(10)}{15 - 10} = 1.517$ or 1.518 cars/min^2	1 : answer
Units of cars/min in (c) and cars/min^2 in (d)	1 : units in (c) and (d)

2002 SCORING GUIDELINES **Question 2**

(a) $\int_9^{17} E(t)\,dt = 6004.270$ 6004 people entered the park by 5 pm.	$3 \begin{cases} 1: \text{limits} \\ 1: \text{integrand} \\ 1: \text{answer} \end{cases}$
(b) $15\int_9^{17} E(t)\,dt + 11\int_{17}^{23} E(t)\,dt = 104048.165$ The amount collected was \$104,048. or $\int_{17}^{23} E(t)\,dt = 1271.283$ 1271 people entered the park between 5 pm and 11 pm, so the amount collected was $\$15 \cdot (6004) + \$11 \cdot (1271) = \$104,041$.	1 : setup
(c) $H'(17) = E(17) - L(17) = -380.281$ There were 3725 people in the park at $t = 17$. The number of people in the park was decreasing at the rate of approximately 380 people/hr at time $t = 17$.	$3 \begin{cases} 1: \text{value of } H'(17) \\ 2: \text{meanings} \\ \quad 1: \text{meaning of } H(17) \\ \quad 1: \text{meaning of } H'(17) \\ \quad < -1 > \text{ if no reference to } t = 17 \end{cases}$
(d) $H'(t) = E(t) - L(t) = 0$ $t = 15.794$ or 15.795	$2 \begin{cases} 1: E(t) - L(t) = 0 \\ 1: \text{answer} \end{cases}$

2001 SCORING GUIDELINES Question 3

(a) Since $v'(2) = a(2)$ and $a(2) = 15 > 0$, the velocity is increasing at $t = 2$.

1 : answer and reason

(b) At time $t = 12$ because

$$v(12) - v(0) = \int_0^{12} a(t)\,dt = 0.$$

$2 : \begin{cases} 1 : t = 12 \\ 1 : \text{reason} \end{cases}$

(c) The absolute maximum velocity is 115 ft/sec at $t = 6$.

The absolute maximum must occur at $t = 6$ or at an endpoint.

$$v(6) = 55 + \int_0^6 a(t)\,dt$$
$$= 55 + 2(15) + \frac{1}{2}(4)(15) = 115 > v(0)$$

$\int_6^{18} a(t)\,dt < 0$ so $v(18) < v(6)$

4 :
- 1 : $t = 6$
- 1 : absolute maximum velocity
- 1 : identifies $t = 6$ and $t = 18$ as candidates
  or
  indicates that v increases, decreases, then increases
- 1 : eliminates $t = 18$

(d) The car's velocity is never equal to 0. The absolute minimum occurs at $t = 16$ where

$$v(16) = 115 + \int_6^{16} a(t)\,dt = 115 - 105 = 10 > 0.$$

$2 : \begin{cases} 1 : \text{answer} \\ 1 : \text{reason} \end{cases}$