§7.1 Integration by Substitution (U-Substitution)

Find the derivatives using the Chain Rule:

1) $\frac{d}{dx}\left(e^{4x^3}\right) = (12x^2)e^{4x^3}$

2) $\frac{d}{dx}\left(\tan\left(2x^5+7x\right)\right) = \sec^2(2x^5+7x)(10x^4+7)$
$= (10x^4+7)\sec^2(2x^5+7x)$

See if you can determine the anti-derivative for these:

3) $\int\left(24x^3\right)\sin\left(6x^4\right) =$
$-\cos(6x^4)+c$

2) $\int\left(\frac{(15x^2+2)}{5x^3+2x+3}\right)dx = \ln\left|5x^3+2x+3\right|+c$

We now will find out how to "*undo*" the Chain Rule using the formal rule called U-Substitution.

Consider the problem of finding $\int(2x+3)\cos\left(x^2+3x\right)dx$

Step 1) Let u = (x^2+3x)

Step 2) Take the derivative with respect to x : $\frac{du}{dx} = 2x+3$

Step 3) Rearrange: $\boxed{du = (2x+3)dx}$

Step 4) Rewrite the integral using substitution: $\int \cos(u)\, du$.

Step 5) Integrate: $\int \cos u\, du = -\sin(u)+c$

Step 6) Substitute back in for u: $-\sin(x^2+3x)+c$

Step 7) Check your answer by taking the derivative of the result in Step 6.

$u = x^2+3x$
$du = (2x+3)dx$

Look for the factor that is a composition of functions... the inner function is most often u.

[Practice Problems]

1. $\int\left(5x^2+1\right)^2(10x)dx$ $u=5x^2+1$ $du=10x\,dx$ $\int u^2 du = \frac{1}{3}u^3+c$ $=\frac{1}{3}(5x^2+1)^3+c$	2. $\int(1+2x)^4(2)dx$ $u=1+2x$ $du=2dx$ $\int u^4 du = \frac{1}{5}u^5+c$ $=\frac{1}{5}(1+2x)^5+c$	3. $\int\left(x^2-1\right)^3(2x)dx$ $u=x^2-1$ $du=2x\,dx$ $\int u^3 dx = \frac{1}{4}u^4+c$ $=\frac{1}{4}(x^2-1)^4+c$

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4. $\int \sqrt{9-x^2}\,(-2x)\,dx$ $u=9-x^2$ $du=-2x\,dx$ $\int\sqrt{u}\,du=\frac{2}{3}x^{3/2}+c$ $=\frac{2}{3}(9-x^2)^{3/2}+c$	5. $\int 2xe^{(x^2+1)}\,dx$ $u=x^2+1$ $du=2x\,dx$ $\int e^u\,du=e^u+c$ $=e^{x^2+1}+c$	6. $\int 4x^3\sqrt{x^4+5}\,dx$ $u=x^4+5$ $du=4x^3\,dx$ $\int\sqrt{u}\,du=\frac{2}{3}x^{3/2}+c$ $=\frac{2}{3}(x^4+5)^{3/2}+c$

The Substitution Rule: If $u=g(x)$ is a differentiable function whose range is an interval I and f is continuous on that interval, then:

$$\int f(g(x))g'(x)dx=\int f(u)du$$

In some cases, we may need to modify $g'(x)$. Consider $\int x\left(x^2+1\right)^2dx$

- Let $u=x^2+1$, then $du=2x\,dx$, what is missing in our integrand?

[More Practice]

7. $\int x^2\sqrt{x^3+1}\,dx$ $u=x^3+1$ $du=3x^2\,dx \quad \frac{1}{3}du=x^2dx$ $\frac{1}{3}\int\sqrt{u}\,du=\frac{2}{9}u^{3/2}+c$ $=\frac{2}{9}(x^3+1)^{\frac{3}{2}}+c$	8. $\int \sec 2x\tan 2x\,dx$ $u=2x$ $du=2dx \quad \frac{1}{2}du=dx$ $\frac{1}{2}\int \sec u\,\cancel{\tan}\,u\,du$ $\frac{1}{2}\sec u+c$ $=\frac{1}{2}\sec(2x)+c$	9. $\int x^3\sqrt{x^4+2}\,dx$ $u=x^4+2$ $du=4x^3dx \quad \frac{1}{4}du=x^3dx$ $\frac{1}{4}\int\sqrt{u}\,du=\frac{1}{6}x^{\frac{3}{2}}+c$ $\frac{1}{6}(x^4+2)^{3/2}+c$
10. $\int\left(1+\frac{1}{t}\right)^2\frac{1}{t^2}\,dt$ $u=1+\frac{1}{t}$ $du=\frac{-1}{t^2} \quad -du=\frac{1}{t^2}dt$ $-\int u^2\,du=\frac{-1}{3}u^3+c$ $=\frac{-1}{3}\left(1+\frac{1}{t}\right)^3+c$	11. $\int\frac{1}{1+(2x)^2}\,dx$ $u=2x$ $du=2dx \quad \frac{1}{2}du=dx$ $\frac{1}{2}\int\frac{1}{1+u^2}du=\frac{1}{2}\arctan u+c$ $=\frac{1}{2}\arctan(2x)+c$	12. $\int\frac{1}{\sqrt{1-9x^2}}\,dx=\int\frac{1}{\sqrt{1-(3x)^2}}dx$ $u=3x$ $du=3dx \quad \frac{1}{3}du=dx$ $\frac{1}{3}\int\frac{1}{\sqrt{1-u^2}}du=\frac{1}{3}\arcsin u+c$ $=\frac{1}{3}\arcsin(3x)+c$
13. $\int x\sin(x^2)\,dx$ $u=x^2$ $du=2x\,dx \quad \frac{1}{2}du=x\,dx$ $\int\sin u\,du=-\cos u+c$ $=-\cos(x^2)+c$	14. $\int\frac{\sin\sqrt{x}}{\sqrt{x}}\,dx$ $u=\sqrt{x}$ $du=\frac{1}{2\sqrt{x}}dx \quad 2du=\frac{dx}{\sqrt{x}}$ $2\int\sin u\,du=-2\cos u+c$ $=-2\cos\sqrt{x}+c$	15. $\int\cos x\sin^2x\,dx=\int(\sin x)^2\cos x\,dx$ $u=\sin x$ $du=\cos x\,dx$ $\int u^2\,du=\frac{1}{3}u^3+c$ $=\frac{1}{3}\sin^3x+c$

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§7.1 More Practice with Integration by Substitution (U-Substitution)

Find the following integrals.

1. $\int e^{-3x}dx$ $u=-3x$ $du=-3dx$ $-\frac{1}{3}du=dx$ $=-\frac{1}{3}\int e^u du=-\frac{1}{3}e^u+c$ $=-\frac{1}{3}e^{-3x}+c$	2. $\int \sin x\cos^3 x dx=\int(\cos x)^3\sin x dx$ $u=\cos x$ $du=-\sin x dx$ $-du=\sin x dx$ $-\int u^3 du=-\frac{1}{4}u^4+c$ $=-\frac{1}{4}\cos^4 x+c$	3. $\int \frac{1}{2x+5}dx$ $u=2x+5$ $du=2dx$ $\frac{1}{2}du=dx$ $\frac{1}{2}\int\frac{1}{u}du=\frac{1}{2}\ln\lvert u\rvert+c$ $=\frac{1}{2}\cdot\ln\lvert 2x+5\rvert+c$
4. $\int \frac{(\ln x)^2}{x}dx$ $u=\ln x$ $du=\frac{1}{x}dx$ $\int u^2 du=\frac{1}{3}u^3+c$ $=\frac{1}{3}(\ln x)^3+c$	5. $\int e^x\sqrt{e^x+5}dx$ $u=e^x+5$ $du=e^x dx$ $\int\sqrt{u}du=\frac{2}{3}u^{3/2}+c$ $=\frac{2}{3}\sqrt{(e^x+5)^3}+c$ $=\frac{2}{3}(e^x+5)^{3/2}+c$	6. $\int \frac{\cos x}{2+\sin x}dx$ $u=2+\sin x$ $du=\cos x dx$ $\int\frac{1}{u}du=\ln\lvert u\rvert+c$ $=\ln\lvert 2+\sin x\rvert+c$
7. $\int \frac{x}{4-x^2}dx$ $u=4-x^2$ $du=-2x dx$ $-\frac{1}{2}du=x dx$ $-\frac{1}{2}\int\frac{1}{u}du=-\frac{1}{2}\ln\lvert u\rvert+c$ $=-\frac{1}{2}\ln\lvert 4-x^2\rvert+c$	8. $\int \frac{e^{\frac{1}{y}}}{y^2}dy$ $u=\frac{1}{y}$ $du=-\frac{1}{y^2}dy$ $-du=\frac{1}{y^2}dy$ $=-\int e^u du=-e^u+c$ $=-e^{\frac{1}{y}}+c$	9. $\int \sin(2\theta+1)d\theta$ $u=2\theta+1$ $du=2d\theta$ $\frac{1}{2}du=d\theta$ $\frac{1}{2}\int\sin(u)du=-\frac{1}{2}\cos u+c$ $=-\frac{1}{2}\cos(2\theta+1)+c$
10. $\int \frac{\tan^2 x}{\cos^2 x}dx=\int(\tan x)^2\sec^2 x dx$ $u=\tan x$ $du=\sec^2 x dx$ $\int u^2 du=\frac{1}{3}u^3+c$ $=\frac{1}{3}\tan^3 x+c$	11. $\int \frac{\ln^3 x}{x}dx$ $u=\ln x$ $du=\frac{1}{x}dx$ $\int u^3 du=\frac{1}{4}u^4+c$ $=\frac{1}{4}\ln^4\lvert x\rvert+c$ $=\frac{1}{4}(\ln\lvert x\rvert)^4+c$ $\frac{1}{4}(\ln x)^4+c$	12. $\int\sqrt{x}(x+2)dx$ distribute $\int x^{3/2}+2x^{1/2}dx$ $=\frac{2}{5}x^{5/2}+\frac{4}{3}x^{3/2}+c$

$\int \tan x\,dx = \ln|\sec x| + c$ $\int \cot x\,dx = \ln|\sin x| + c$ Ch. 7 KEY p.4

13. * $\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx$

$u = \cos x$
$du = -\sin x\,dx$
$-du = \sin x\,dx$
$-\int \frac{1}{u}\,du = -\ln|u| + c$
$= -\ln|\cos x| + c$
$= \ln|\sec x| + c$

14. * $\int \cot x\,dx = \int \frac{\cos x}{\sin x}\,dx$

$u = \sin x$
$du = \cos x\,dx$
$\int \frac{1}{u}\,du = \ln|u| + c$
$= \ln|\sin x| + c$

15. $\int (3\tan^5 x)\sec^2 x\,dx$

$u = \tan x$
$du = \sec^2 x\,dx$
$3\int u^5\,du = \frac{3}{6}u^6 + c$
$= \frac{1}{2}u^6 + c$
$= \frac{1}{2}\tan^6 x + c$

16. $\int (8\cot^3 x)\csc^2 x\,dx$

$u = \cot x$
$du = -\csc^2 x\,dx$
$-du = \csc^2 x\,dx$
$-8\int u^3\,du = -2u^4 + c$
$= -2\cot^4 x + c$

17. $\int \frac{\cos(3x)}{\sin^8(3x)}\,dx$

$u = \sin(3x)$
$du = 3\cos(3x)\,dx$
$\frac{1}{3}du = \cos(3x)\,dx$
$\frac{1}{3}\int \frac{1}{u^8}\,du = \frac{1}{3}\int u^{-8}\,du$
$= \frac{1}{21}u^{-7} + c$
$= \frac{1}{21\sin^7(3x)} + c$

18. FIX

$\int \csc(6x^2+3x)\cot(6x^2+3x)(4x+1)\,dx$

$u = 6x^2+3x$
$du = 12x+3 = 3(4x+1)\,dx$
$\frac{1}{3}du = (4x+1)\,dx$
$\frac{1}{3}\int \csc(u)\cot(u)\,du = \frac{-1}{3}\csc u + c$
$= -\frac{1}{3}\csc(6x^2+3x) + c$

19. $\int \left(4^{3x^5+9x}\right)(5x^4+3)\,dx$

$u = 3x^5+9x$
$du = 15x^4+9 = 3(5x^4+3)\,dx$
$\frac{1}{3}du = (5x^4+3)\,dx$
$\frac{1}{3}\int 4^u\,du = \left(\frac{1}{3\ln 4}\right)(4^u) + c$
$= \left(\frac{1}{3\ln 4}\right)\cdot 4^{3x^5+9x} + c$
$= \frac{4^{3x^5+9x}}{\ln 48} + c$

20. $\int e^{5x^2+35x}(2x+7)\,dx$ ← FIX

$u = 5x^2+35x$
$du = (10x+35)\,dx$
$du = 5(2x+7)\,dx$
$\frac{1}{5}du = (2x+7)\,dx$
$\frac{1}{5}\int e^u\,du = \frac{1}{5}e^u + c$
$\frac{1}{5}e^{(5x^2+35x)} + c$

21. $\int \sec(3x^4+8x)\tan(3x^4+8x)(3x^3+2)\,dx$

$u = 3x^4+8x$
$du = 12x^3+8 = 4(3x^3+2)\,dx$
$\frac{1}{4}du = (3x^3+2)\,dx$
$\frac{1}{4}\int \sec(u)\tan(u)\,dx$
$= \frac{1}{4}\sec u + c$
$= \frac{1}{4}\sec(3x^4+8x) + c$

22. $\int \cos(5^{3x})(5^{3x})\,dx$

$u = 5^{3x}$
$du = 5^{3x}\cdot 3\ln 5\,dx$
$\frac{1}{3\ln 5}du = 5^{3x}\,dx$
$\frac{1}{3\ln 5}\int \cos u\,du = \frac{\sin u}{3\ln 5} + c$
$= \frac{1}{3\ln 5}\cdot \sin(5^{3x}) + c$
$= \frac{\sin(5^{3x})}{\ln(125)} + c$

23. $\int \sec^2(\ln x^7)\left(\frac{1}{x}\right)dx$

$u = \ln(x^7) = 7\ln x$
$du = \frac{7}{x}\,dx$
$\frac{1}{7}du = \frac{dx}{x}$
$\frac{1}{7}\int \sec^2 u\,du = \frac{1}{7}\tan u + c$
$= \frac{1}{7}\tan(\ln(x^7)) + c$

24. $\int \csc^2(7x^3-6x^2)(7x^2-4x)\,dx$

$u = 7x^3-6x^2$
$du = 21x^2-12x\,dx$
$\frac{1}{3}du = (7x^2-4x)\,dx$
$\frac{1}{3}\int \csc^2(u)\,du$
$= -\frac{1}{3}\cot(u) + c$
$= \frac{-1}{3}\cot(7x^3-6x^2) + c$

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method:

1. $\int_0^1 \frac{x^2}{1+x^3}\,dx$

$u = 1+x^3$
$du = 3x^2\,dx$
$\frac{1}{3}du = x^2\,dx$

u-limits
$$\boxed{\begin{aligned} u_1 &= 1+1^3 = 2 \\ u_0 &= 1+0^3 = 1 \end{aligned}}$$

$$\frac{1}{3}\int_1^2 \frac{1}{u}\,du = \frac{1}{3}\left(\ln|u|\right)\Big|_1^2$$
$$= \frac{1}{3}\left(\ln|2| - \ln|1|\right)$$
$$= \frac{1}{3}\ln|2| = \frac{1}{3}\ln 2$$

2. $\int_1^3 \frac{(\ln z)^2}{z}\,dz$

$u = \ln z$
$du = \frac{1}{z}dz$

u-limits
$$\boxed{\begin{aligned} u_3 &= \ln|3| = \ln|3| \\ u_1 &= \ln|1| = 0 \end{aligned}}$$

$$\int_0^{\ln|3|} (u^2)\,du$$
$$\frac{1}{3}u^3\Big|_0^{\ln|3|} = \frac{1}{3}\left((\ln|3|)^3 - 0^3\right)$$
$$= \frac{1}{3}(\ln 3)^3$$

3. $\int_1^5 x\sqrt{x^2-1}\,dx$

$u = x^2 - 1$
$du = 2x\,dx$

u-limits
$$\boxed{\begin{aligned} u_5 &= 24 \\ u_1 &= 0 \end{aligned}}$$

$$\frac{1}{2}\int_0^{24} \sqrt{u}\,du = \frac{1}{2}\cdot\frac{2}{3}\; u^{3/2}\Big|_0^{24}$$
$$= \frac{1}{3}\left((24)^{3/2} - 0\right)$$
$$= \frac{1}{3}(24)^{3/2} \leftarrow \text{ok}$$
$$= \frac{1}{3}(4\cdot 6)^{3/2} = \frac{8}{3}(6)^{3/2}$$

4. $\int_0^1 x(x^2+1)^3\,dx$

$u = x^2+1$
$du = 2x\,dx$

u-limits
$$\boxed{\begin{aligned} u_1 &= 2 \\ u_0 &= 1 \end{aligned}}$$

$$\frac{1}{2}\int_1^2 u^3\,du = \frac{1}{8}u^4\Big|_1^2$$
$$= \frac{1}{8}(2^4 - 1^4)$$
$$= \frac{15}{8}$$

5. $\int_0^1 (3x-1)^3\,dx$

$u = 3x-1$
$du = 3\,dx$

u-limits
$$\boxed{\begin{aligned} u_1 &= 2 \\ u_0 &= -1 \end{aligned}}$$

$$\frac{1}{3}\int_{-1}^2 u^3\,du = \frac{1}{12}u^4\Big|_{-1}^2$$
$$\frac{1}{12}\left(2^4 - (-1)^4\right)$$
$$= \frac{15}{12} = \frac{5}{4}$$

6. $\int_0^1 x\sqrt{1-x^2}\,dx$

$u = 1-x^2$
$du = -2x\,dx$
$-\frac{1}{2}du = x\,dx$

u-limits
$$\boxed{\begin{aligned} u_1 &= 0 \\ u_0 &= 1 \end{aligned}}$$

$$-\frac{1}{2}\int_1^0 \sqrt{u}\,du = \frac{1}{2}\int_0^1 \sqrt{u}\,du$$
$$\frac{1}{3}\cdot u^{3/2}\Big|_0^1 = \frac{1}{3}(1-0) = \frac{1}{3}$$