The Fundamental Theorem of Calculus - The Integral Function – class exploration
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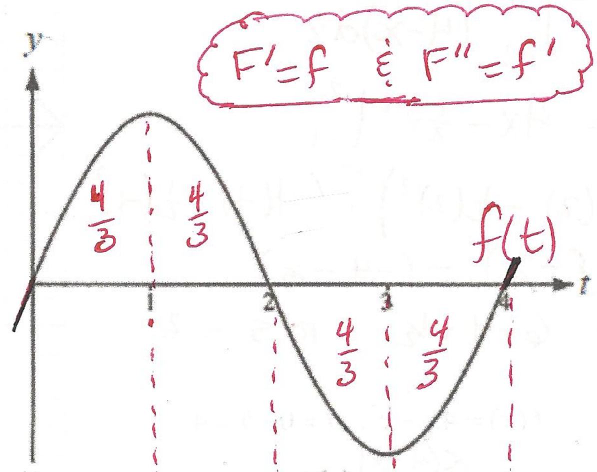
ALWAYS WRITE THESE !

$\boxed{F' = f \ \& \ F'' = f'}$

The graph of the function $y = f(t)$ is shown. The function is defined for $0 \le t \le 4$ and has the following properties:

- The graph of f has *odd symmetry* around the point $(2,0)$.
- On the interval $0 \le t \le 2$, the graph of f is symmetric with respect to the line $t = 1$.
- $\int_0^1 f(t)\,dt = \frac{4}{3}$



Graph of $y = f(t)$

F(x) Behavior

$F' = f > 0$ F is INC.	$F' = f < 0$ F is DEC.

$F'' = f > 0$ F ccup	$F'' = f < 0$ F ccdown	$F'' = f > 0$ F ccup

1. Let $F(x) = \int_0^x f(t)\,dt$

a. Complete the table.

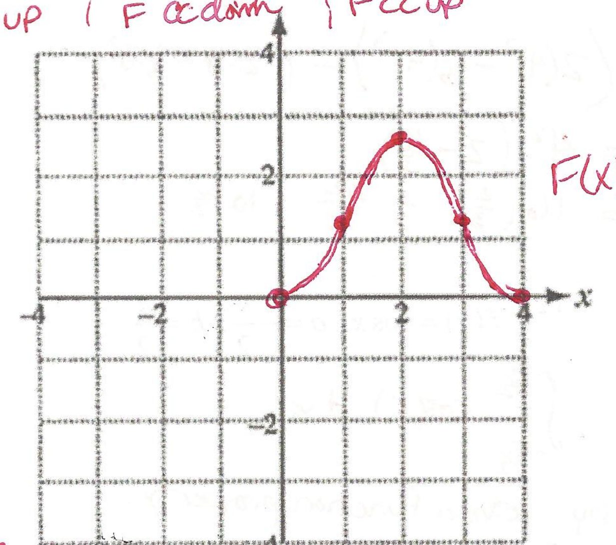
x	0	1	2	3	4
$F(x)$	0	4/3	8/3	4/3	0

b. Plot the points from your table on the coordinate grid.

Before you sketch the curve consider what you know about the behavior of the graph of $F(x)$ based on the graph of $f(t)$. (Behavior: increasing/decreasing and concavity.)

↳ see chart above

Now sketch the graph of $F(x)$.



$F' = f > 0$ on $(0,2)$ ∴ $F(x)$ is increasing
$F' = f < 0$ on $(2,4)$ ∴ $F(x)$ is decreasing

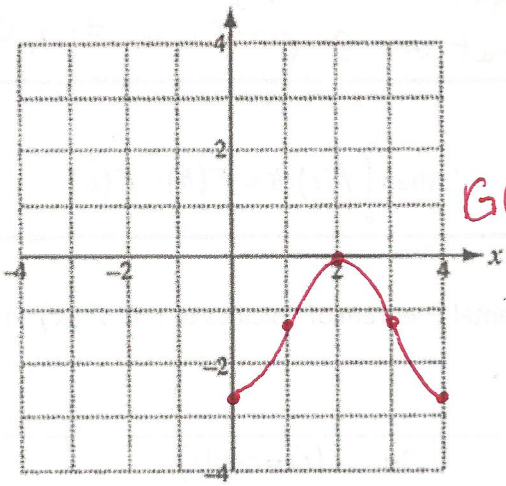
$F' = f$ is increasing on $(0,1)$ & $(3,4)$ ∴ $F'' = f' > 0$ & F is concave up.
$F' = f$ is decreasing on $(1,3)$ ∴ $F'' = f' < 0$ & F is concave down.

2. Let $G(x)=\int_{2}^{x} f(t)\,dt$

a. Complete the table.

x	0	1	2	3	4
$F(x)$	$-8/3$	$-4/3$	0	$-4/3$	$-8/3$

b. Plot the points from your table on the coordinate grid. Sketch the graph of $G(x)$.



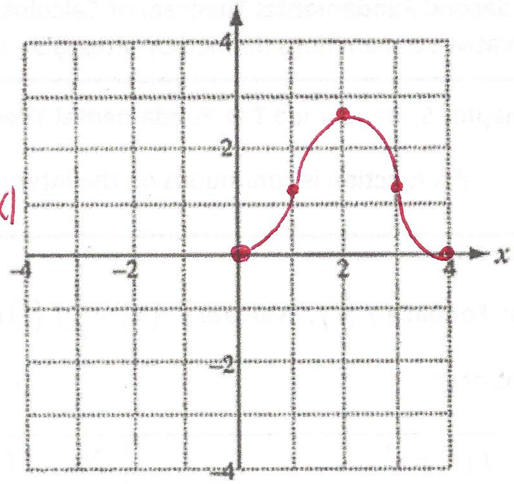
SAME BEHAVIOR for $G(x)$ & $H(x)$ as for $F(x)$

3. Let $H(x)=\int_{4}^{x} f(t)\,dt$

a. Complete the table.

x	0	1	2	3	4
$F(x)$	0	$4/3$	$8/3$	$4/3$	0

b. Plot the points from your table on the coordinate grid. Sketch the graph of $H(x)$.



4. Complete the following table.

	$F(x)$	$G(x)$	$H(x)$
The maximum value of the function occurs at what x-value?	$x=2$	$x=2$	$x=2$
Justify your conclusion using Calculus. MAXIMUM occurs at $x=2$ b/c $F'=f$ changes signs positive to negative.			
The minimum value of the function occurs at what x-value?	$x=0,4$	$x=0,4$	$x=0,4$
Justify your conclusion using Calculus. MINIMUM occurs at $x=0$ & 4 b/c $F'=f$ changes signs negative to positive.			
The function increases on what intervals of x?	$(0,2)$	$(0,2)$	$(0,2)$
Justify your conclusion using Calculus. $F(x)$ ($G(x)$ & $H(x)$) are increasing on $(0,2)$ b/c $F'=f>0$			
The function decreases on what intervals of x?	$(2,4)$	$(2,4)$	$(2,4)$
Justify your conclusion using Calculus. $F(x)$ ($G(x)$ & $H(x)$) are decreasing on $(2,4)$ b/c $F'=f<0$			
The function is concave up on what intervals of x?	$(0,1),(3,4)$	$(0,1),(3,4)$	$(0,1),(3,4)$
Justify your conclusion using Calculus. $F(x)$ ($G(x)$ & $H(x)$) are concave up on $(0,1),(3,4)$ b/c $F'=f$ is inc $\therefore F''=f'>0$.			
The function is concave down on what intervals of x?	$(1,3)$	$(1,3)$	$(1,3)$
Justify your conclusion using Calculus. $F(x)$ ($G(x)$ & $H(x)$) are concave down on $(1,3)$ b/c $F'=f$ is dec $\therefore F''=f'<0$.			

5. What conjectures would you make about the family of functions of the form $W(x)=\int_k^x f(t)\,dt$ for $0\le k\le 4$, where f is the graph given at the beginning of the worksheet?

- All of the graphs in this family of functions have the same behavior in terms of inc/dec & ccup/ccdown
- They will be vertical translations (shift down or up) compared to the graphs of F(x), G(x) & H(x) that we graphed in #1, 2, & 3.

The Second Fundamental Theorem of Calculus

Derivatives of Functions defined by Integrals – class exploration

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In chapter 5, we learned **The Fundamental Theorem of Calculus:**

If a function is continuous on the interval from $[a,b]$ and $f=F'$, then $\int_a^b f(t)\,dt = F(b)-F(a)$.

#1-3: For each $f(t)$, evaluate $F(x)=\int_1^x f(t)\,dt$ using the Fundamental Theorem of Calculus to find $F(x)$ in terms of x.

1a. $f(t)=t^3$	2a. $f(t)=4t-t^2$	3a. $f(t)=\cos(t)$
$F(x)=\int_1^x t^3\,dt$ $F(x)=\frac{1}{4}t^4\Big\vert_1^x$ $F(x)=\frac{1}{4}x^4-\frac{1}{4}$	$F(x)=\int_1^x 4t-t^2\,dt$ $F(x)=2t^2-\frac{1}{3}t^3\Big\vert_1^x$ $F(x)=\left(2x^2-\frac{1}{3}x^3\right)-\left(2-\frac{1}{3}\right)$ $F(x)=2x^2-\frac{1}{3}x^3-\frac{5}{3}$	$F(x)=\int_1^x \cos(t)\,dt$ $F(x)=\sin t\Big\vert_1^x$ $F(x)=\sin x-\sin(1)$

Next take the derivative of each of the $F(x)$ functions you found in part 1a, 2a, and 3a above.

1b.	2b.	3b.
$F'(x)=x^3+0$ $F'(x)=x^3$	$F'(x)=4x-x^2+0$ $F'(x)=4x-x^2$	$F'(x)=\cos x+0$ $F'(x)=\cos x$

Let's reconsider #1-3 from above with a new definition: $F(x)=\int_{1}^{x^2} f(t)\,dt$.

Repeat the process. Use the Fundamental Theorem of Calculus to find $F(x)$ in terms of x.

1c. $f(t)=t^3$ $F(x)=\int_{1}^{\sin x} t^3\,dt$ $F(x)=\frac{1}{4}t^4\Big\vert_1^{\sin x}$ $F(x)=\frac{1}{4}(\sin x)^4-\frac{1}{4}$	2c. $f(t)=4t-t^2$ $F(x)=\int_{1}^{\sin x} 4t-t^2\,dt$ $F(x)=2t^2-\frac{1}{3}t^3\Big\vert_1^{\sin x}$ $F(x)=\left(2\sin^2 x-\frac{1}{3}\sin^3 x\right)-\left(2-\frac{1}{3}\right)$ $=2\sin^2 x-\frac{1}{3}\sin^3 x-\frac{5}{3}$	3c. $f(t)=\frac{1}{t}$ $F(x)=\int_{2}^{\sin x} \frac{1}{t}\,dt$ $F(x)=\ln\lvert t\rvert\,\Big\vert_1^{\sin x}$ $F(x)=\ln(\sin x)-\ln(2)$

Take the derivative of each of the $F(x)$ functions you found in part 1c, 2c, and 3c above.

1d. $F'(x)=(\sin x)^3\cdot\cos x+0$ $F'(x)=\sin^3 x\cdot(\cos x)$	2d. $F'(x)=4(\sin x)\cos x-(\sin x)^2\cdot\cos x$ $F'(x)=(4\sin x-\sin^2 x)\cdot\cos x$	3d. $F'(x)=\frac{1}{\sin x}(\cos x)+0$ $F'(x)=\frac{1}{\sin x}\cdot(\cos x)$

In summary:

2nd FTC 1: The Second Fundamental Theorem of Calculus part 1

If $F(x)=\int_{a}^{x} f(t)\,dt$ where a is a constant and f is a continuous function,

then $F'(x)=f(x)$.

2nd FTC 2: The Second Fundamental Theorem of Calculus part 2

If $F(x)=\int_{a}^{g(x)} f(t)\,dt$ where a is a constant and f is a continuous function, and g is a differentiable function

then $F'(x)=f(g(x))\cdot g'(x)$.

Can you use the **Second Fundamental Theorem of Calculus** to find the following derivatives <u>without</u> going through the process of anti-deriving and then deriving?

4a. $F(x)=\int_1^x 7\sqrt{t}\,dt$ $F'(x)=$ $7\sqrt{x}$	5a. $F(x)=\int_1^x \tan(t)\,dt$ $F'(x)=$ $\tan(x)$	6a. $F(x)=\int_1^x \frac{1}{\sqrt[3]{t}}\,dt$ $F'(x)=$ $\frac{1}{\sqrt[3]{x}}$
4b. $F(x)=\int_3^{\tan x} 7\sqrt{t}\,dt$ $F'(x)=$ $7\sqrt{\tan x}\cdot(\sec^2 x)$	5b. $F(x)=\int_3^{3x^2+5x} \tan(t)\,dt$ $F'(x)=$ $\tan(3x^2+5x)(6x+5)$	6b. $F(x)=\int_3^{e^x} \frac{1}{\sqrt[3]{t}}\,dt$ $F'(x)=$ $\frac{1}{\sqrt[3]{e^x}}\cdot e^x$
4c. $F(x)=\int_2^{g(x)} 7\sqrt{t}\,dt$ $F'(x)=$ $7\sqrt{g(x)}\cdot g'(x)$	5b. $F(x)=\int_2^{h(x)} \tan(t)\,dt$ $F'(x)=$ $\tan(h(x))\cdot h'(x)$	6c. $F(x)=\int_2^{w(x)} \frac{1}{\sqrt[3]{t}}\,dt$ $F'(x)=$ $\frac{1}{\sqrt[3]{w(x)}}\cdot w'(x)$

7. Let $H(x)=\int_{\frac{\pi}{2}}^{x} t\cos(t)\,dt$ for $0<x<2\pi$.

ORGANIZE YOUR THOUGHTS...

	QI	QII QIII	QIV
cos x	+	−	+
x	+	+	+

$0 \nearrow \frac{\pi}{2} \searrow \frac{3\pi}{2} \nearrow 2\pi$

a. Determine the critical number of $H(x)$.

$H'(x) = x\cos(x)$

$H'(x) = 0$ when $x = 0$ & $\cos x = 0$

$x = \pi/2,\ 3\pi/2$

b. Determine which critical numbers correspond to a relative maximum value of $H(x)$. Justify your answer.

$H(x)$ has a Rel Maximum at $x = \pi/2$ b/c

$H'(x)$ changes signs from positive to negative.

c. Determine which critical numbers correspond to a relative minimum value of $H(x)$. Justify your answer.

$H(x)$ has a Rel minimum at $x = 3\pi/2$ b/c

$H'(x)$ changes signs from negative to positive.

§6.4 FTC Part II – More Practice

STATEMENT OF THE FUNDAMENTAL THEOREM OF CALCULUS PART II
If a function f is continuous on [a,b], then the function $F(x)=\int_a^x f(t)\,dt$ has a derivative at every point in (a,b) and $F'(x)=\frac{d}{dx}\left[\int_a^x f(t)\,dt\right]=f(x)$
Also note: If $F(x)=\int_a^{g(x)} f(t)\,dt$ then $F'(x)=\frac{d}{dx}\left[\int_a^{g(x)} f(t)\,dt\right]=f(g(x))\cdot g'(x)$

If $f(x)=\int_1^x t^2\,dt$

then $f'(x)=$

If $f(x)=\int_1^{x^2} t^2\,dt$

then $f'(x)=$

Two important things to keep in mind:

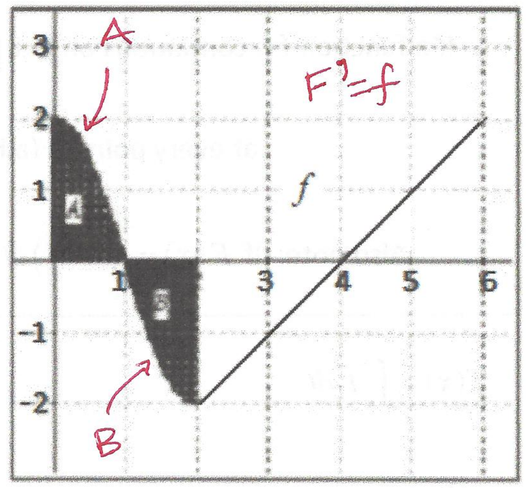
- *Always make the lower limit the constant and the upper limit the variable*
- *Multiple the answer plugged in by the derivative of the limit*

Practice: Find the derivative of each of the following functions.

1. $g(x)=\int_{-2}^{x}\sqrt{1+4t^2}\,dt$ $g'(x)=\sqrt{1+4x^2}$	2. $y=\int_1^x (1+t)^5\,dt$ $y'=(1+x)^5$	3. $F(x)=\int_0^x 3\,dt$ $F'(x)=3$
4. $F(x)=\int_2^x \sin t\,dt$ $F'(x)=\sin x$	5. $F(x)=\int_x^1 \tan t\,dt$ $F(x)=-\int_1^x \tan t\,dt$ $F'(x)=-\tan x$	6. $F(x)=\int_x^2 \cos(t^2)\,dt$ $F(x)=-\int_2^x \cos(t^2)\,dt$ $F'(x)=-\cos(x^2)$
7. $F(x)=\int_2^{3x} 2t\,dt$ $F'(x)=2(3x)\cdot 3=18x$	8. $F(x)=\int_\pi^{\sin x} 2t\,dt$ $F'(x)=2(\sin x)(\cos x)=\sin(2x)$	9. $F(x)=\int_a^x f(t)\,dt$ $F'(x)=f(x)$
10. $F(x)=\int_c^{g(x)} f(t)\,dt$ $F'(x)=f(g(x))\cdot g'(x)$	11. $h(x)=\int_0^{x^2}\sqrt{1+r^3}\,dt$ $h'(x)=\sqrt{1+x^6}\,(2x)$	12. $y=\int_{e^x}^0 \sin^3 t\,dt$ $y=-\int_0^{e^x}\sin^3(t)\,dx$ $y=-\sin^3(e^x)$
13. $h(x)=\int_1^{1/x}\ln t\,dt$ $h'(x)=\ln\left(\frac{1}{x}\right)\cdot\left(\frac{-1}{x^2}\right)$	14. $g(u)=\int_{1-4u^2}^{-1}\cos t\,dt$ $g(u)=-\int_{-1}^{1-4u^2}\cos t\,dt$ $g'(u)=-\cos(1-4u^2)(-8u)$ $=+8u\cos(1-4u^2)$	

Curriculum Module: Calculus: Functions Defined by Integrals

Let $F(x)=\int_1^{2x} f(t)\,dt$, where the graph of f on the interval $0 \le t \le 6$ is shown at the right, and the regions A and B each have an area of 1.3.



a. Compute $F(0)$ & $F(1)$.

$F(0) = \int_1^0 f(t)\,dt = -\int_0^1 f(t)\,dt = \underline{-1.3}$

$F(1) = \int_1^2 f(t)\,dt = \underline{-1.3}$

b. Determine $F'(x) = f(2x)\cdot 2$

$F'(x) = 2f(2x)$

c. Determine the critical numbers of $F(x)$ on the interval $0 \le t \le 3$

ORGANIZE YOUR THOUGHTS.

$F'(x) = 0$

$2f(2x) = 0$

$f(2x) = 0$

$2x \in \{1, 4\} \quad \therefore x \in \{\tfrac{1}{2}, 2\}$

+ 0 − 0 +

↗ $\tfrac{1}{2}$ ↘ 2 ↗

d. Determine which critical numbers of $F(x)$ corresponds to a maximum value of $F(x)$ on the interval $0 \le t \le 3$. Justify your answer.

$F(x)$ has a relative maximum at $x = \tfrac{1}{2}$

b/c $F' = f$ changes signs from positive to negative

e. Determine which critical numbers of $F(x)$ corresponds to a minimum value of $F(x)$ on the interval $0 \le t \le 3$. Justify your answer.

$F(x)$ has a relative minimum at $x = 2$

b/c $F' = f$ changes signs from negative to positive