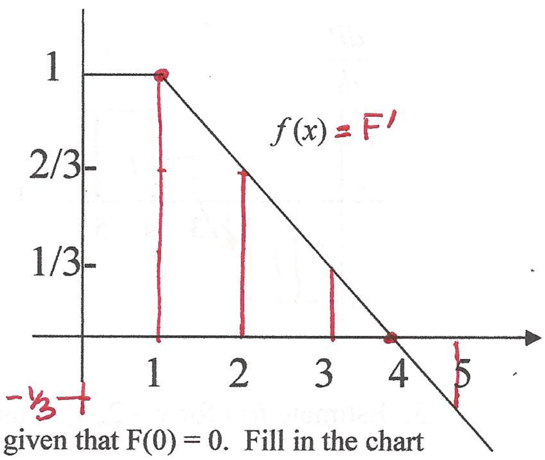
NOTES KEY

AP Calculus AB—Unit 6

§6.1 & §6.2 Anti-derivatives: Graphically, Numerically and Analytically

1. The graph of f is given in Figure 6.1. Please note that $f = F'$.



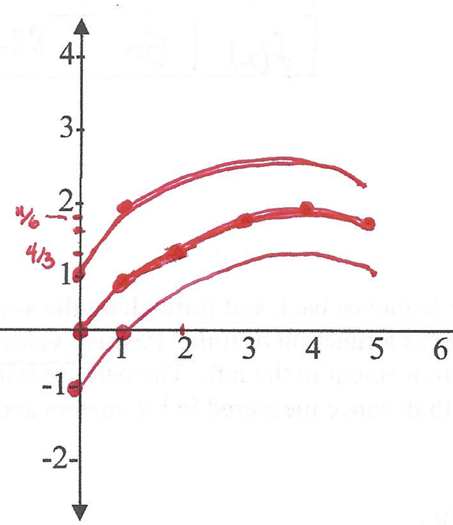
   a. What are the critical points of F(x)?
   when $F' = 0$ $x = 4$

   b. Identify local extrema.
   @ $x = 4$ $F' = f$ changes signs ⊕ to ⊖ ∴ rel max.

   c. Discuss concavity and points of inflection.
   $f' = F''$ f is decreasing on $(1,5)$ ∴ $f' = F'' < 0$ & F is concave down

   d. Sketch an accurate graph of F, the anti-derivative of f. You are given that F(0) = 0. Fill in the chart using the Fundamental Theorem of Calculus: $\int_a^b f(x)dx = F(b) - F(a)$. Then sketch F.

x	$\int_a^b f(x)dx$	$F(x)$
0	$\int_0^0 f(x)dx$	$0 + 0 = 0$
1	$\int_0^1 f(x)dx$	$0 + 1 = 1$
$\frac{1}{2}(1)(1+\frac{2}{3})$ 2	$\int_0^2 f(x)dx$	$0 + 1 + \frac{1}{3} = \frac{4}{3}$
$\frac{1}{2}(1)(\frac{2}{3}+\frac{1}{3})$ 3	$\int_0^3 f(x)dx$	$0 + 1 + \frac{1}{3} + \frac{1}{2} = \frac{11}{6}$
$\frac{1}{2}(1)(\frac{1}{3})$ 4	$\int_0^4 f(x)dx$	$0 + 1 + \frac{1}{3} + \frac{1}{2} + \frac{1}{6} = 2$
5	$\int_0^5 f(x)dx$	$0 + 1 + \frac{1}{3} + \frac{1}{2} + \frac{1}{6} - \frac{1}{6} = \frac{11}{6}$



e. On the same axes, now sketch F but given that $F(0) = 1$.

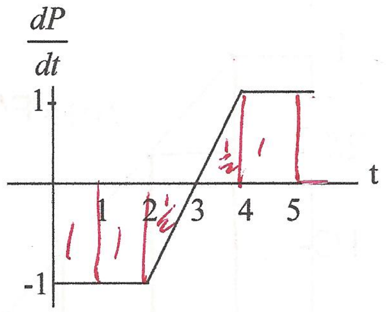
x	$F(x)$
0	1
1	2
2	$\frac{7}{3}$
3	$\frac{17}{6}$
4	3
5	$\frac{17}{6}$

Graphs are vertically shifted from the original

f. Repeat again but now $F(0) = -1$

x	$F(x)$
0	-1
1	0
2	$\frac{1}{3}$
3	$\frac{5}{6}$
4	1
5	$\frac{5}{6}$

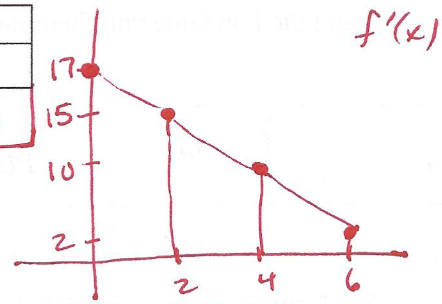
2. Use the figure below and the fact that $P = 2$ when $t = 0$ to find values of P when $t = 1, 2, 3, 4$ and 5.



x	$P(x)$
0	2
1	1
2	0
3	$-\frac{1}{2}$
4	0
5	1

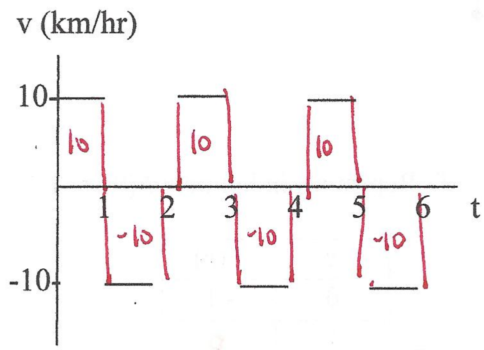
3. Estimate $f(x)$ for $x = 2, 4, 6$, using the given values of $f'(x)$ and the fact that $f(0) = 50$.

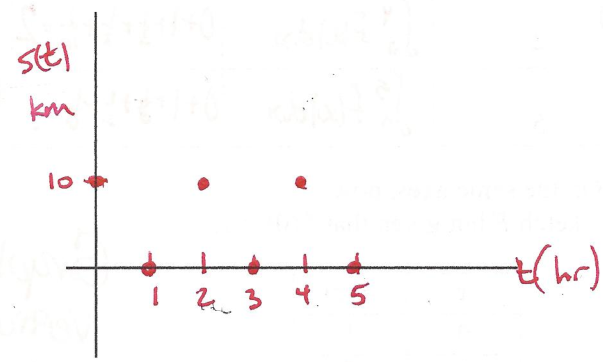
x	0	2	4	6
$f'(x)$	17	15	10	2
$f(x)$	50	82	107	123

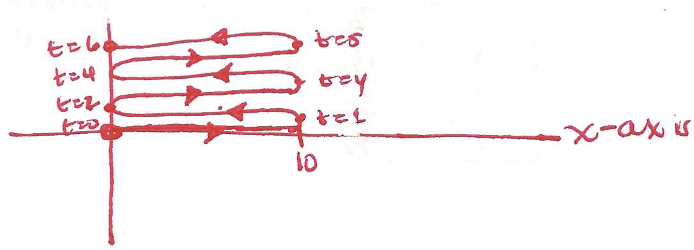


$\frac{1}{2}(2)(17+15)$ $\frac{1}{2}(2)(15+10)$ $\frac{1}{2}(2)(10+6)$
32 25 16

4. A particle moves back and forth along the x-axis. The figure below (left one) approximates the velocity of the particle as a function of time. Positive velocities represent movement to the right and negative velocities represent movement to the left. The particle starts at the point (5, 0). Graph the distance of the particle from the origin, with distance measured in kilometers and time in hours (LABEL!!!).







AP Calculus AB—Unit 6

What is a function that has a derivative of $3x^2$?

A function is an **antiderivative** of f on an open interval I if $F'(x) = f(x)$ for all x in I.

There is an entire family of antiderivatives for each derivative. The **general** antiderivative of $3x^2$ is $x^3 + c$. c is called the **constant of integration**.

We use the following symbols: $\int f(x)dx = F(x) + c$ → Constant of Integration

Antiderivative

Integral Symbol

Integrand

Variable of Integration

Basic Rules for constructing Antiderivatives Analytically:

1. Zero $\quad \int 0\,dx = c$
2. Constant $\quad \int k\,dx = kx + C$
3. Power Rule $\quad \int x^n dx = \dfrac{x^{n+1}}{n+1} + C$
4. Natural Log $\quad \int \dfrac{1}{x} dx = \ln|x| + C$
5. Exponential $\quad \int e^x dx = e^x + C$
6. General Rules $\quad \int kf(x)dx = k\int f(x)dx$

$$\int (f(x) \pm g(x))dx = \int f(x)dx \pm \int g(x)dx$$

Example: Find the following:

a) $\int x^5 dx = \frac{1}{6}x^6 + c$

b) $\int x^{37} dx = \frac{1}{38}x^{38} + c$

c) $\int x^{-2} dx = -1x^{-1} + c$

$\frac{-1}{x} + c$

d) $\int x^{\frac{1}{2}} dx = \frac{2}{3}x^{3/2} + c$

e) $\int \frac{1}{x^3} dx$

$\int x^{-3}dx = \frac{-1}{2}x^{-2}$

$= \frac{-1}{2x^2} + c$

f) $\int \sqrt[3]{x}dx = \int x^{1/3}dx$

$\frac{3}{4}x^{4/3} + c$

g) $\int (2x^2+1)dx$

$\frac{2}{3}x^3 + x + c$

h) $\int (1-x^3)dx$

$x - \frac{1}{4}x^4 + c$

i) $\int \left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)dx$

$\int x^{\frac{1}{2}} - x^{-\frac{1}{2}}\,dx$

$\frac{2}{3}x^{3/2} - 2x^{\frac{1}{2}} + c$

j) $\int \sqrt{x}(x^2+1)dx$

$\int x^{5/2} + x^{\frac{1}{2}}\,dx$

$\frac{2}{7}x^{7/2} + \frac{2}{3}x^{3/2} + c$

k) $\int (x^2+1)(x+2)dx$

FOIL

$\int x^3 + 2x^2 + x + 2\,dy$

$\frac{1}{4}x^4 + \frac{2}{3}x^3 + \frac{1}{2}x^2 + 2x + c$

l) $\int \left(\sqrt{x} + \frac{1}{2\sqrt{x}}\right)dx = \int x^{\frac{1}{2}} + \frac{1}{2}x^{-\frac{1}{2}}\,dx$

$\frac{2}{3}x^{3/2} + 1x^{\frac{1}{2}} + c$

m) $\int \frac{x^3-x^2+1}{x}dx$

$\int x^2 - x + \frac{1}{x}\,dx$

$\frac{1}{3}x^3 - \frac{1}{2}x^2 + \ln|x| + c$

n) $\int \frac{4x^5-3x^2-5}{x^2}dx$

$\int 4x^3 - 3 - 5x^{-2}\,dx$

$x^4 - 3x + 5x^{-1} + c$

$x^4 - 3x + \frac{5}{x} + c$

o) $\int \frac{2x-x^2+3}{\sqrt{x}}dx = \int 2x^{\frac{1}{2}} - x^{3/2} + 3x^{-\frac{1}{2}}\,dx$

$\frac{4}{3}x^{3/2} - \frac{2}{5}x^{5/2} + 6x^{\frac{1}{2}} + c$

n) $\int 8dx = 8x + c$

o) $\int 8xdx = 4x^2 + c$

p) $\int 8x^2dx$ $\frac{8}{3}x^3 + c$

q) $\int 8x^3dx = 2x^4 + c$

r) $\int \pi x^3dx = \frac{\pi}{4}x^4 + c$

s) $\int \frac{e^x}{5}dx = \frac{1}{5}e^x + c$

t) $\int (3x+x^2)dx$

$= \frac{3}{2}x^2 + \frac{1}{3}x^3 + c$

1u) $\int \frac{1}{2x}dx = \frac{1}{2}\int \frac{1}{x}\,dy$

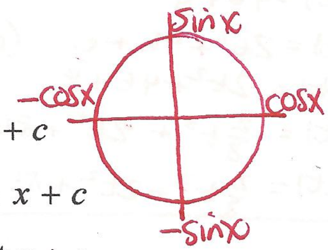
$\frac{1}{2}\ln|x| + c$

v) $\int (\frac{1}{x} + \frac{1}{x^3})dx = \ln|x| - \frac{1}{2x^2} + c$

x^{-3}

§6.1 – §6.2 Antiderivatives—cont.

Trig Rules:



1. $\int \cos x\,dx = \sin x + c$
2. $\int \sin x\,dx = -\cos x + c$
3. $\int \sec^2 x\,dx = \tan x + c$
4. $\int \sec x \tan x\,dx = \sec x + c$
5. $\int \csc^2 x\,dx = -\cot x + c$
6. $\int \csc x \cot x\,dx = -\csc x + c$

1. Find the following:

a) $\int (2\cos x - \sin x)dx$

$2\sin x + \cos x + c$

b) $\int 2\sec x \tan x\,dx$

$2\sec x + c$

c) $\int -3\csc^2 x\,dx$

$3\cot x + c$

d) $\int \frac{1}{2}\csc x \cot x\,dx$

$-\frac{1}{2}\csc x + c$

e) $\int \sec x(\sec x + \tan x)dx$

$\int \sec^2 x + \sec x \tan x\,dx$

$\tan x + \sec x + c$

f) $\int \frac{1}{\csc x}dx = \int \sin x\,dx$

$-\cos x + c$

g) $\int (3x^2 - 5x + \sin x)dx$

$x^3 - \frac{5}{2}x^2 - \cos x + c$

h) $\int (e^x - \sec^2 x - \frac{5}{x})dx$

$e^x - \tan x - 5\ln|x| + c$

i) $\int (\sin^2 x + \cos^2 x)dx$

$\int (1)\,dx$

$x + c$

j) $\int (10x^4 - 2\sec^2 x)\,dx$

$2x^5 - 2\tan x + c$

k) $\int \frac{\cos\theta}{\sin^2\theta}d\theta = \int \cot x \csc x\,dx$

$= -\csc x + c$

2. Initial Conditions: You can solve for a particular c if you are given certain preliminary information; find $\int (2x+3)dx$ such that $F(1) = 4$.

$y = x^2 + 3x + c \quad (1,4)$

$4 = 1^2 + 3(1) + c$

$4 = 4 + c$

$\therefore c = 0$

$\underline{y = x^2 + 3x + 0}$

3. Find the velocity function $v(t)$ and position function $s(t)$ corresponding to the acceleration function $a(t)=4t+4$ given $v(0)=8$ and $s(0)=12$.

$a(t)=4t+4$
$v(t)=2t^2+4t+c$ $(0,8)$ $8=0+0+c$
$v(t)=2t^2+4t+8$ $c=8$
$s(t)=\frac{2}{3}t^3+2t^2+8t+c$ $(0,12)$
$s(t)=\frac{2}{3}t^3+2t^2+8t+12$ $c=12$

Other Rules:

Box THIS

1. $\int a^x dx=\frac{a^x}{\ln a}+c$ 2. $\int \frac{1}{\sqrt{1-x^2}}dx=\arcsin x+c$ 3. $\int \frac{1}{1+x^2}dx=\arctan x+c$

2. Find the following

a) $\int\left(x^2-3x+2^x\right)dx$
$\frac{1}{3}x^3-\frac{3}{2}x^2+\frac{2^x}{\ln 2}+c$

b) $\int\left(\frac{1}{3^{-x}}-x^3\right)dx$ 3^x-x^3
$\frac{3^x}{\ln 3}-\frac{1}{4}x^4+c$

c) $\int\frac{2}{\sqrt{1-x^2}}dx$
$2\arcsin(x)+c$

d) $\int\left(\frac{1}{2x^2}-\frac{3}{1+x^2}\right)dx$
$\int\left(\frac{1}{2}x^{-2}-3\frac{1}{1+x^2}\right)dx$
$-\frac{1}{2x}-3\arctan x+c$

e) $\int\left(x^{\frac{2}{3}}-4x^{\frac{-1}{5}}+\frac{4}{x}\right)dx$
$\frac{3x^{5/3}}{5}-5x^{4/5}+4\ln|x|+c$

f) $\int\left(\frac{2}{x}+3e^x\right)dx$
$2\ln|x|+3e^x+c$

g) $\int\left(\frac{1}{2t}-\sqrt{2}e^t\right)dt$
$\frac{1}{2}\ln|t|-\sqrt{2}e^t+c$

h) $\int(4\sin x+3\cos x)dx$
$-4\cos x+3\sin x+c$

i) $\int\left(\frac{1}{\csc x}\right)dx=\int \sin x\,dx$
$=-\cos x+c$

j) $\int\left(4\sec^2 x+\csc x\cot x\right)dx$
$4\tan x-\csc x+c$

k) $\int \sec x(\sec x+\tan x)dx$
$\int(\sec^2 x+\sec x\tan x)\,dx$
$=\tan x+\sec x+c$

l) $\int\frac{1-x+3\sqrt{x}}{x^2}dx$
$\int\left(\frac{1}{x^2}-\frac{1}{x}+\frac{3}{x^{3/2}}\right)dx$
$\int x^{-2}-\frac{1}{x}+3x^{-3/2}\,dx$
$=-\frac{1}{x}-\ln|x|+\frac{-6}{\sqrt{x}}+c$

m) $\int \sec x(\sec x+\cos x)dx$
$\int(\sec^2 x+1)\,dx$
$\tan x+x+c$

n) $\int\frac{\sin x}{\cos^2 x}dx$ $\int \tan x\cdot\sec x\,dx$
$=\sec x+c$

o) $\int\frac{\sin 2x}{\cos x}dx=\int\frac{2\sin x\cos x}{\cos x}dx$
$=\int 2\sin x\,dx$
$=-2\cos x+c$

change * p) $\int\left(2^x-x^2+\frac{1}{4^x}\right)dx$ $\int 4^{-x}dx$
$\frac{2^x}{\ln 2}-\frac{1}{3}x^3$

q) $\int\left(e^x-\frac{1}{1+x^2}\right)dx$
$e^x-\arctan x+c$

r) $\int\left(3^x+\frac{1}{\sqrt{1-x^2}}\right)dx$
$\frac{3^x}{\ln 3}+\arcsin x+c$

$\frac{d}{dx}(4^{-x})=(-\ln 4)(4^{-x})=\frac{-\ln 4}{4^x}$

§6.2—cont. Evaluating the Definite Integral

By the FTC part 1, we know the following: $\int_a^b f(x)dx = F(b) - F(a)$ where $f = F'$

Example 1: Evaluate $\int_1^5 2x\,dx$ by hand.

By FTC, we know $\int_1^5 2x\,dx = F(5) - F(1)$. Since $F(x) = \int 2x\,dx = x^2 + C$ we can conclude the following:

$F(x) = x^2 + C$

$F(5) = 5^2 + C$
$F(1) = 1^2 + C$

$F(5) - F(1) = (5^2 + C) - (1^2 + C) = 24$

Therefore $\int_1^5 2x\,dx = F(5) - F(1) = 24$

Example 2: Evaluate $\int_0^{\frac{5\pi}{6}} \cos x\,dx$ by hand. (*Notice the simpler notation*)

$$\int_0^{\frac{5\pi}{6}} \cos x\,dx = \sin x \Big|_0^{\frac{5\pi}{6}} = \sin(\tfrac{5\pi}{6}) - \sin(0) = \frac{1}{2} - 0 = \frac{1}{2}$$ Use your calculator to check this answer.

Practice: Evaluate the following using proper notation shown in Example 2. Check your answers with your calculator.

1. $\int_0^1 x^3 dx$

$\frac{1}{4}x^4 \Big|_0^1 = \frac{1}{4}(1^4 - 0^4)$
$= \frac{1}{4}$

2. $\int_{-2}^1 5dx$

$5x \Big|_{-2}^1$
$5(1 - -2)$
$5(3) = 15$

3. $\int_{-2}^4 (\frac{x}{2} + 3)dx$

$\frac{1}{4}x^2 + 3x \Big|_{-2}^4$
$[\frac{1}{4}(4)^2 + 3(4)] - [\frac{1}{4}(-2)^2 + 3(-2)]$
$(4+12) - (1-6) = 21$

4. $\int_{0.5}^{1.5} (-2x + 4)dx$

$-x^2 + 4x \Big|_{\frac{1}{2}}^{\frac{3}{2}}$
$[-(\frac{3}{2})^2 + 4(\frac{3}{2})] - [-(\frac{1}{2})^2 + 4(\frac{1}{2})]$
$(-\frac{9}{4} + 6) - (-\frac{1}{4} + 2)$
$-\frac{9}{4} + \frac{1}{4} + 6 - 2 = -2 + 4 = 2$

5. $\int_0^1 x\,dx$

$\frac{1}{2}x^2 \Big|_0^1$
$= \frac{1}{2}(1^2 - 0)$
$= \frac{1}{2}$

6. $\int_0^2 (1 - x^2)dx$

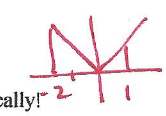
$x - \frac{1}{3}x^3 \Big|_0^2$
$(2 - \frac{1}{3}(2)^3) - (0)$
$2 - \frac{8}{3}$
$\frac{6}{3} - \frac{8}{3} = -\frac{2}{3}$

7. $\int_0^1 e^x dx$

$e^x \Big|_0^1$
$= e^1 - e^0$
$= e - 1$

8. $\int_{-2}^1 |x| dx$

Think Geometrically!



$\int_{-2}^0 (-x)\,dx + \int_0^1 x\,dx$
$-\frac{1}{2}x^2 \Big|_{-2}^0 + \frac{1}{2}x^2 \Big|_0^1$
$0 - (-\frac{1}{2}(-2)^2) + (\frac{1}{2}(1)^2 - 0)$
$2 + \frac{1}{2}$
$= \frac{5}{2}$

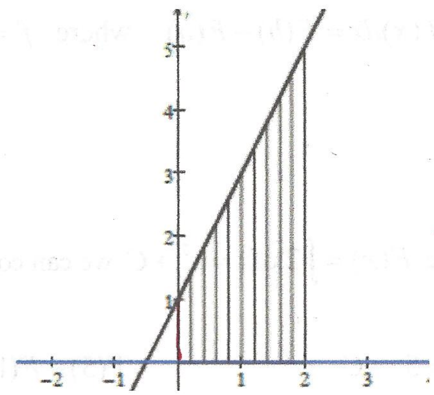
AP Calculus AB—Unit 6

For each of the following problems, write a definite integral to express the area under the curve. The equation for the pictured function is provided. Then use the Fundamental Theorem of Calculus to find the area.

6. $f(x)=2x+1$

Geometrically

$\frac{1}{2}(2)(1+5)=6$



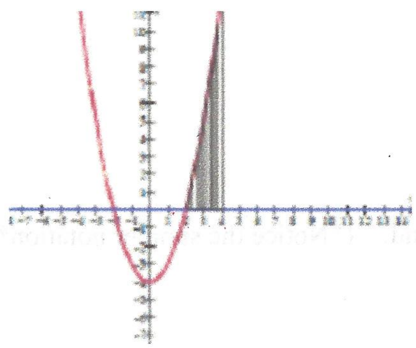
$\int_0^2 (2x+1)\,dx =$

$= x^2+x \Big|_0^2$

$= (2^2+2)-(0^2+0)$

$= 6$

7. $f(x)=x^2-4$



$\int_2^4 (x^2-4)\,dx$

$\frac{1}{3}x^3-4x \Big|_2^4$

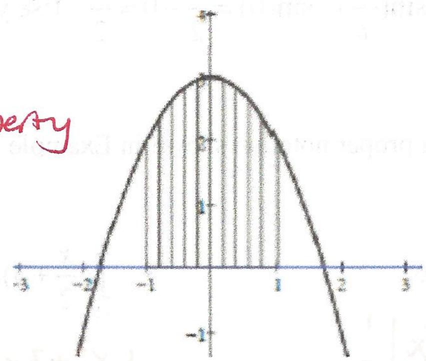
$\left(\frac{1}{3}(4)^3-4(4)\right)-\left(\frac{1}{3}(2)^3-4(2)\right)$

$4^2\left(\frac{1}{3}(4)-1\right)-8\left(\frac{1}{3}-1\right)$

$\frac{16}{3}+\frac{16}{3}=\frac{32}{3}=10\frac{2}{3}$ ✓

8. $f(x)=3-x^2$

even function property



$2\int_0^1 (3-x^2)\,dx$

$2\left(3x-\frac{1}{3}x^3\right)\Big|_0^1$

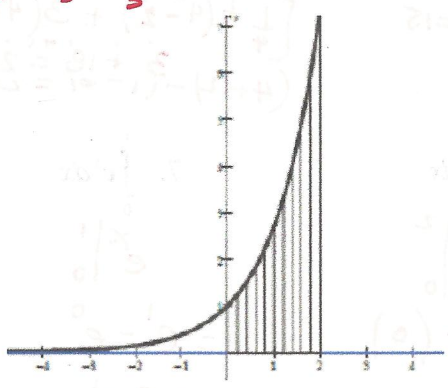
$2\left(\left(3-\frac{1}{3}\right)-(0)\right)=2\left(\frac{8}{3}\right)=\frac{16}{3}$ ✓

$\int_{-1}^1 (3-x^2)\,dx$

$3x-\frac{1}{3}x^3 \Big|_{-1}^1$

$\left(3-\frac{1}{3}\right)-\left(-3+\frac{1}{3}\right)=6-\frac{2}{3}=\frac{16}{3}$ ✓

9. $f(x)=e^x$



$\int_0^2 e^x\,dx$

$e^x \Big|_0^2 = e^2-e^0$

$= e^2-1$

≈ 6.389

For each of the following, graph and find the area under the graph of $f(x)$ from a to b. Show the definite integral and your evaluation.

10. $f(x) = 4 - x, \ a = -1, \ b = 2$

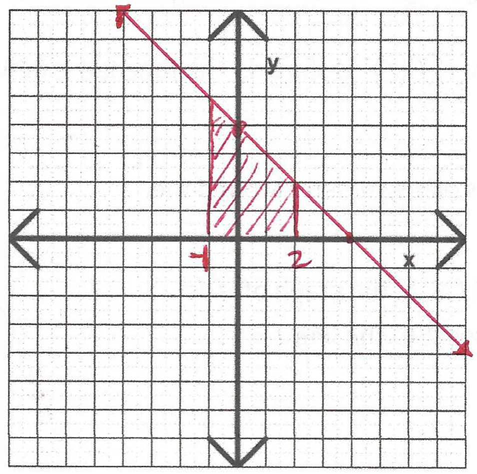
$\int_{-1}^{2} (4-x)dx$

$= 4x - \frac{1}{2}x^2 \Big|_{-1}^{2}$

$= \left(4(2) - \frac{1}{2}(2)^2\right) - \left(4(-1) - \frac{1}{2}(-1)^2\right)$

$= (8-2) - (-4 - \frac{1}{2})$

$= 6 + 4 + \frac{1}{2} = 10.5 = \frac{21}{2}$



$y=4-x$

Geometrically:

$\frac{1}{2}(3)(5+2)$

$= \frac{21}{2}$

11. $f(x) = 4x - x^2, \ a = 0, \ b = 4$

$x(4-x)$

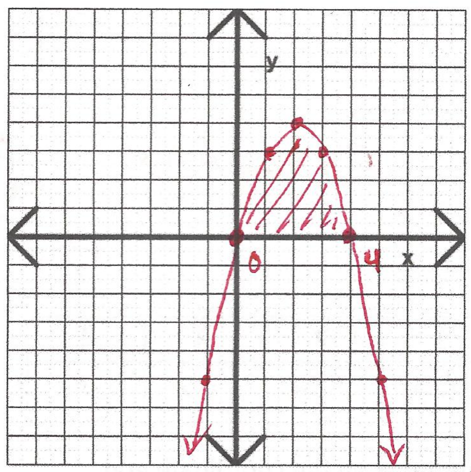
$\int_{0}^{4} 4x - x^2 \, dx$

$= 2x^2 - \frac{1}{3}x^3 \Big|_{0}^{4}$

$= \left(2(4)^2 - \frac{1}{3}(4)^3\right) - \cancel{\left(2\cdot 0 - \frac{1}{3}\cdot 0\right)}$

$= 4^2\left(2 - \frac{4}{3}\right)$

$= 16\left(\frac{2}{3}\right) = \frac{32}{3} = 10\frac{2}{3}$



$y=4x-x^2$

12. $f(x) = \cos x, \ a = -\frac{\pi}{2}, \ b = \frac{\pi}{2}$

$\int_{-\pi/2}^{\pi/2} \cos(x) \, dx$

by even function property.

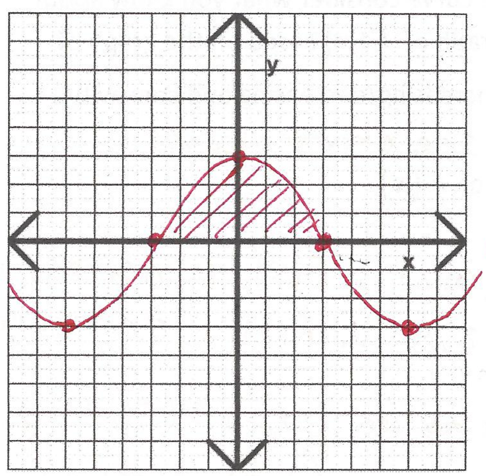
$= 2\int_{0}^{\pi/2} \cos(x) \, dx$

$= 2 \sin(x) \Big|_{0}^{\pi/2}$

$= 2\left(\sin \frac{\pi}{2} - \sin 0\right)$

$= 2(1 - 0)$

$= 2$



$y = \cos x$