

§5.3 Interpretations of the Definite Integral – Student Notes

The **Total Change Theorem**: The integral of a rate of change is the **total change**:

$$\int_a^b F'(x)dx = F(b) - F(a)$$

This theorem is used in many applications.

Example 1: A bacteria population which is 5 million at time $t = 0$, is growing at an instantaneous rate of 2^t million bacteria per hour. Estimate the total increase in the bacteria population during the first hour, and the population at $t = 1$.

Example 2: A deer population is increasing at a rate of $\frac{dp}{dt} = 20 + 25t$ per year (where t is measured in years). By how much does the deer population increase between the 4th and 10th years? Show integral set up and answer.

Example 3: A particle moves along a line so that its velocity at time t is $v(t) = t^2 - t - 6$ (measured in meters per second). Find the displacement of the particle during the time period $1 \leq t \leq 4$. Find the distance traveled during this time period.

{Note: For velocity functions $\int_{t_1}^{t_2} v(t)dt = s(t_2) - s(t_1)$, this represents the change in position or

displacement. For total distance traveled, you need the total area or $\int_{t_1}^{t_2} |v(t)| dt$. }

Average Value of a Function Lab

(Created by Ms. Haller – Adapted from MAA Materials Lab 13)

Warmup:

1. What do you usually do when you want to find the average of a set of data values?
2. Do that with this data set: 90, 100, 85, 75, 93.
3. What would it mean to find the average value of a function?
 - Consider the function $y = x$ on the interval $[0, 5]$.
 - What do you think the average of ALL the y -values on this graph will be? Why?

Playing with blocks:

4. Build a model of $y = x$ on $[0, 5]$ with the blocks.
 - What is the average of the data set: 0, 1, 2, 3, 4, 5?
 - Rearrange the blocks into a **rectangle** that has the same base length as your original model.
 - What is the height of this rectangle?
 - How does it compare to the average value you found in question #3?
 - Depending on your answer: why are they the same OR why are they different?

Important Definitions:

$$\text{Mid-range value of } f \text{ over } [a, b] = \frac{\max f(x) + \min f(x)}{2}.$$

$$\text{Average value of } f \text{ over } [a, b] = \frac{1}{b-a} \int_a^b f(x) dx.$$

Calculate both the mid-range value and the average value for $f(x) = x$. You will have to “think backwards” for the anti-derivative function you need for the average value computation.

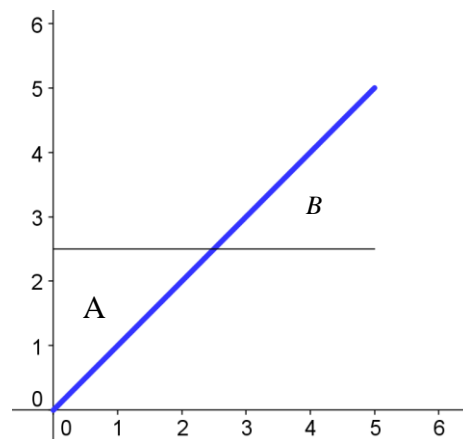
5. Mid-range value of $f(x) = x$ over $[0, 5] =$

6. Average value of $f(x) = x$ over $[0, 5] =$

Let’s look at this graphically:

Notice the horizontal line at the level of the **average value** on the graph.

7. Show or explain how you know that the area in region A is the same as the area in region B.



Playing with blocks:

8. Build a model of $y = x^2$ on $[0, 3]$ with the blocks.
 - What is the average of the data set: 0, 1, 4, 9?
 - Rearrange the blocks into a **rectangle** that has the same base length as your original model.
 - What is the height of this rectangle?

Calculate both the mid-range value and the average value for $f(x) = x^2$. You will have to “think backwards” for the anti-

derivative function you need for the average value computation.

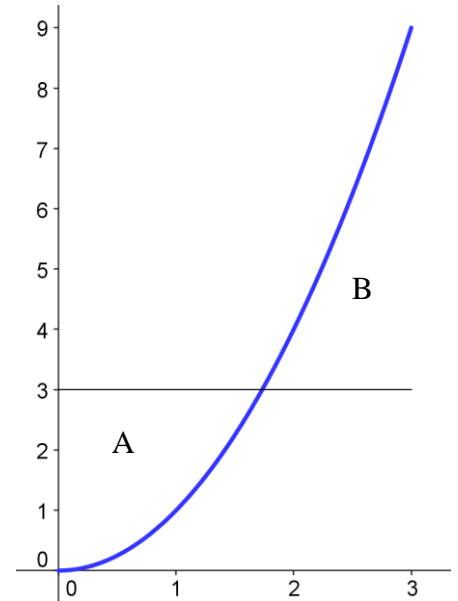
9. Mid-range value of $f(x) = x^2$ over $[0, 3] =$

10. Average value of $f(x) = x^2$ over $[0, 3] =$

Let's look at this graphically:

11. Draw a horizontal line at the level of the average value on the graph of f over the interval $[0, 3]$.

12. Use calculus to show that the area of region A is the same as the area of region B.



13. A simpler method is to show that

$$\int_0^3 (f(x) - \text{average value of } f) dx = 0.$$

- Try this.

- Why does this method work? Hint: What does the graph of $f(x) - \text{average value of } f$ look like?

Average vs. mid-range

In the problems in this lab you noticed that the mid-range value was likely to be different than the average value of the function. In this question, we will explore whether there is anything about the shape of the graphs of the functions that could account for this difference.

The graph of $f(x) = e^x$ is concave up on $[0, 1]$. You will have to “think backwards” for the anti-derivative function you need for the average value computation.

14. Mid-range value of $f(x) = e^x$ over $[0, 1] =$	15. Average value of $f(x) = e^x$ over $[0, 1] =$	16. Is the mid-range above or below the average value of $f(x)$?
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The graph of $f(x) = \sin x$ is concave down on $[0, \frac{\pi}{2}]$.

17. Mid-range value of $f(x) = \sin x$ over $[0, \frac{\pi}{2}] =$	18. Average value of $f(x) = \sin x$ over $[0, \frac{\pi}{2}] =$	19. Is the mid-range above or below the average value of $f(x)$?
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20. State a conjecture regarding concavity, mid-range, and average value of a function for a continuous function over the interval $[a, b]$.

21. Prove this conjecture. The trapezoid rule (see Easton for more information) will help you get started.

- Area of a trapezoid = $\frac{(b_1 + b_2) \cdot h}{2}$.

A new look at an old problem:

“Suppose a stone is thrown upward from a 54 foot high platform with an initial velocity of 30 feet per second. After t seconds have elapsed, its velocity is $v(t) = 30 - 32t$. It lands on the ground 3 seconds after it was thrown.”

22. If the stone begins 54 feet high and lands on the ground, what is the *change in position* of the stone? Please pay attention to the sign of this value.

23. Earlier this year, we defined average velocity to be: $\frac{\text{change in position}}{\text{change in time}}$.

- Calculate the average velocity of the stone on $[0, 3]$ using this definition.

24. Now, use the new definition for average value of a function to determine the average value of the velocity function on $[0, 3]$.

25. Use the Fundamental Theorem of Calculus to explain the equality of your answers in parts a and b.

26. Why did Ms. Haller have you play with blocks?

- a. She knew you would like it.
- b. To help you visually understand “averaging”.
- c. To make you think about the difference between discrete and continuous functions.
- d. All of the above.

Definite Integral as an Average

Think of finding the average of temperatures during the day in your room at n equally spaced times, t_1, t_2, \dots, t_n . How do you compute the average of the temperatures algebraically?

$$\text{Average temperature} \approx \frac{f(t_1) + f(t_2) + f(t_3) + \dots + f(t_n)}{n}$$

The larger we make n , the better the approximation. We can rewrite this expression as a Riemann sum over the interval $0 \leq t \leq 30$ if we use the fact that $\Delta t = \frac{30}{n}$, so $n = \frac{30}{\Delta t}$:

$$\begin{aligned} \text{Average temperature} &\approx \frac{f(t_1) + f(t_2) + f(t_3) + \dots + f(t_n)}{\left(\frac{30}{\Delta t}\right)} \\ &= \frac{f(t_1)\Delta t + f(t_2)\Delta t + f(t_3)\Delta t + \dots + f(t_n)\Delta t}{30} \\ &= \frac{1}{30} \sum_{i=1}^n f(t_i)\Delta t \end{aligned}$$

As $n \rightarrow \infty$, the Riemann sum tends towards an integral, and $1/30$ of the sum also approximates the average temperature better.

$$\text{Average temperature} = \lim_{n \rightarrow \infty} \frac{1}{30} \sum_{i=1}^n f(t_i)\Delta t = \frac{1}{30} \int_0^{30} f(t) dt$$

Average Value of $f(x)$

on interval from $x = a$ to $x = b$: $\frac{1}{b-a} \int_a^b f(x) dx$

See Doug Kuhlmann's Geogebra sketch: <https://www.geogebra.org/m/tbcGQmz5>

To visualize the Average Value graphically using the Geogebra sketch, try the functions

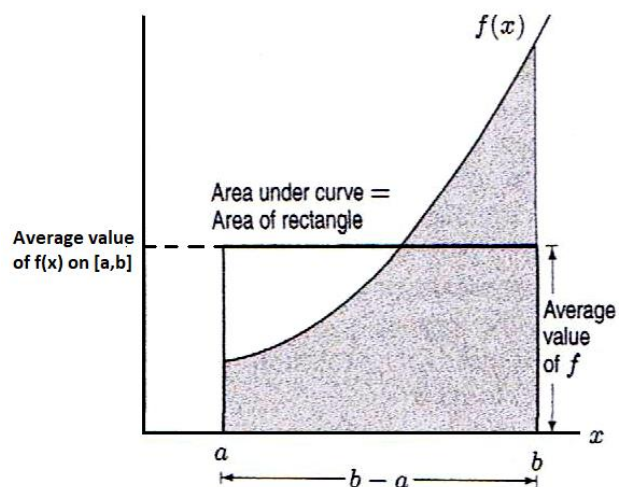
$$f(x) = 4 - 2x$$

$$f(x) = 4 - x^2$$

$$f(x) = x^3 + 2x^2$$

How to Visualize the Average on a Graph:

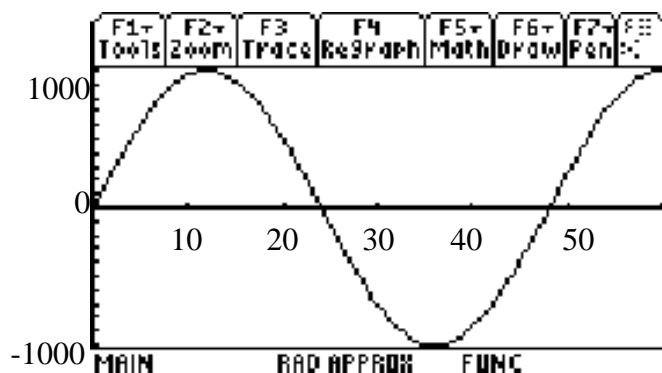
$$(\text{Average value of } f) \cdot (b - a) = \int_a^b f(x) dx$$



Example 4: An oil spill and its subsequent clean up was monitored. It was determined that the rate of change of the area affected by the spill could be modeled by:

$$R(t) = 1000 \sin\left(\frac{\pi}{24}t\right)$$

where R is measure in ft^2/hr and time, t , is measured in hours. The graph below is of the function $R(t)$.



- Interpret $\int_0^{10} R(t)dt$.
- When will the oil spill be at a maximum?
- When was the oil spill clean up making progress?
- Find the average rate at which the oil is spilling out over the time period 0 to 12 hours.

Example 5. (Calc) A news broadcast in early 1993 said the average American's annual income is changing at a rate of $r(t) = 40(1.002)^t$ dollars per month, where t is in months from January 1, 1993. How much did the average American's income change during 1993?

Example 6. (Calc) If the population in Mexico is modeled by the function $P(t) = 67.38(1.026)^t$ where P is in millions and t is in years since 1980, predict the average population of Mexico between the years of 2000 and 2020. What are the units? Show set up.

Example 7. Suppose $C(t)$ represents the daily cost of heating your house, measured in dollars per day, where t is time in days and $t = 0$ corresponds to January 1, 1997.

- a. Explain what the following represents. What are the units?

$$\int_0^{90} C(t) dt$$

- b. Explain what the following represents. What are the units?

$$\frac{1}{90-0} \int_0^{90} C(t) dt$$

PRACTICE: Interpret the Meanings of Definite Integral Expressions

- 1) $C(t)$ is the amount of coffee (in ounces) in a cup at time t , in minutes:

$$\frac{1}{6-0} \int_0^6 C(t) dt = 10.1$$

- 2) $W(t)$ is the temperature of water (in °F) in a tub at time t , in minutes:

$$\frac{1}{20-0} \int_0^{20} W(t) dt = 60.79$$

- 3) Johanna's velocity $v(t)$ is measured in meters per minute. Interpret:

$$a) \int_0^{40} |v(t)| dt = 7600$$

$$b) \int_0^{40} v(t) dt = 5840$$

- 4) If $f(t)$ is the amount of cola consumed in the U.S. in billions of gallons per year and $t = 0$ corresponds to the year 1990, interpret the meaning of

$$a) \int_0^{16} f(t) dt = 9$$

$$b) \frac{1}{16-0} \int_0^{16} f(t) dt = 0.5625$$

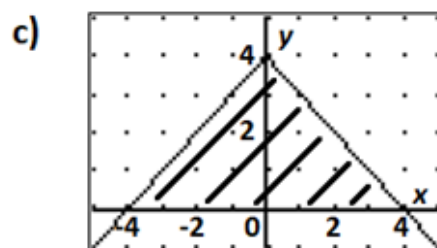
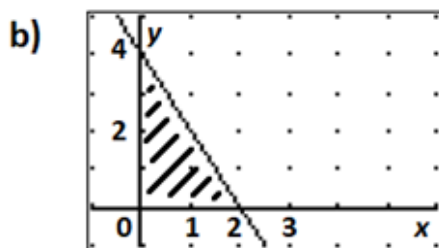
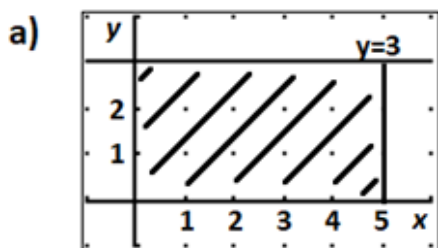
Example 8: Sketch the region corresponding to each definite integral. Then evaluate each integral using a geometric formula.

a) $\int_1^3 4 dx =$

b) $\int_0^3 (x + 2) dx =$

c) $\int_{-2}^2 \sqrt{4 - x^2} dx =$

Example 9: Set up a definite integral that yields the area of the region then, evaluate the integral.



Example 10: For the graph $f(x)$ given to the right find the following geometrically.

a. $\int_{-1}^3 f(x) dx =$ _____

b. $\int_0^3 f(x) dx =$ _____

c. average value over $[-1, 3]$

d. $f'(1) =$ _____

e. $f''(0) =$ _____

f. average value over $[0, 3]$

