KEY

$\int_a^b v(t)dt = s(b) - s(a)$ DAY 86

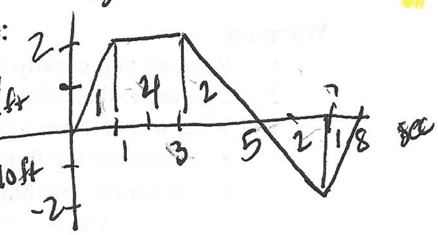
The **Total Change Theorem**: The integral of a rate of change is the **total change**:

$$\int_a^b F'(x)dx = F(b) - F(a)$$

This theorem is used in many applications.

disp $= \int_0^8 v(t)dt = 4$ ft

total dist $\int_0^8 |v(t)|dt = 10$ ft



Example 1: A bacteria population which is 5 million at time $t = 0$, is growing at an instantaneous rate of 2^t million bacteria per hour. Estimate the total increase in the bacteria population during the first hour, and the population at $t = 1$.

Total change in population of bacteria in the 1st hour $= \int_0^1 2^t dt = 1.442695$ million $= 1.442$ or 1.443 million.

Population at time $t=1$ $= P(0) + \int_0^1 2^t dt = 5 + 1.442$
$= 6.442$ or 6.443 million.

Example 2: A deer population is increasing at a rate of $\frac{dp}{dt} = 20 + 25t$ per year (where t is measured in years). By how much does the deer population increase between the 4th and 10th years? Show integral set up and answer.

Total change in deer population between year 4 & year 10. $= \int_4^{10} (20+25t)\,dt = 1170$

Deer population increases by 1170 btwn year 4 & 10.

$\int_4^{10} \frac{dP}{dt}\, dt =$

$\int_4^{10} P'(t)\, dt =$

Example 3: A particle moves along a line so that its velocity at time t is $v(t) = t^2 - t - 6$ (measured in meters per second). Find the displacement of the particle during the time period $1 \le t \le 4$. Find the distance traveled during this time period.

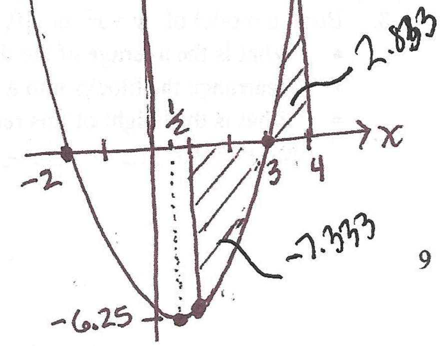
$(t-3)(t+2)$

{Note: For velocity functions $\int_{t_1}^{t_2} v(t)dt = s(t_2) - s(t_1)$, this represents the change in position or displacement. For total distance traveled, you need the total area or $\int_{t_1}^{t_2} |v(t)|\,dt$.}

Displacement $= \int_1^4 (t^2 - t - 6)\,dt = -4.5$ meters

The particle is displaced 4.5 meters to the left.

Total Distance Traveled $= \int_1^4 |t^2 - t - 6|\,dt = 10.166$ meters
10.167 meters



NOTE: $\int_1^3 v(t)\,dt = -7.333$

$\int_3^4 v(t)\,dt = 2.833$

DAY 90

Average Value of a Function Lab

(Created by Ms. Haller – Adapted from MAA Materials Lab 13)

Warmup:

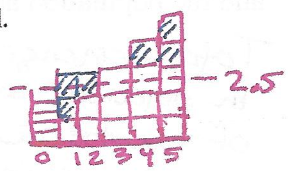
1. What do you usually do when you want to find the average of a set of data values?
2. Do that with this data set: 90, 100, 85, 75, 93.
3. What would it mean to find the average value of a function?
   - Consider the function $y = x$ on the interval [0, 5]. → $\frac{0+1+2+3+4+5}{6} = \frac{15}{6} = \frac{5}{2} = 2.5$
   - What do you think the average of ALL the y-values on this graph will be? Why?

   The same as the average of 1st + last ⇒ $\frac{0+5}{2} = 2.5$

Playing with blocks:

4. Build a model of $y = x$ on [0, 5] with the blocks.
   - What is the average of the data set: 0, 1, 2, 3, 4, 5?
   - Rearrange the blocks into a **rectangle** that has the same base length as your original model.
   - What is the height of this rectangle? 3
   - How does it compare to the average value you found in question #3? 0.5 more.
   - Depending on your answer: why are they the same OR why are they different?

   Discrete blocks – can't divide in half.



Important Definitions:

$$\text{Mid-range value of } f \text{ over } [a, b] = \frac{\max f(x) + \min f(x)}{2}.$$

$$\text{Average value of } f \text{ over } [a, b] = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

Calculate both the mid-range value and the average value for $f(x) = x$. You will have to "think backwards" for the anti-derivative function you need for the average value computation.

5. Mid-range value of $f(x) = x$ over [0, 5] = $\frac{0+5}{2} = 2.5$

CALCULATOR

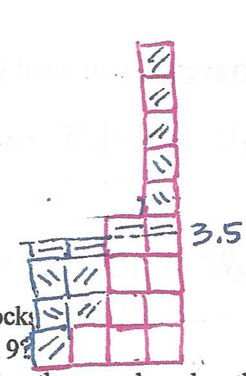
6. Average value of $f(x) = x$ over [0, 5] = $\frac{1}{5-0}\int_0^5 x\,dx = 2.5$ MATH+9: $\int_0^5 x\,dx = \frac{25}{2}$

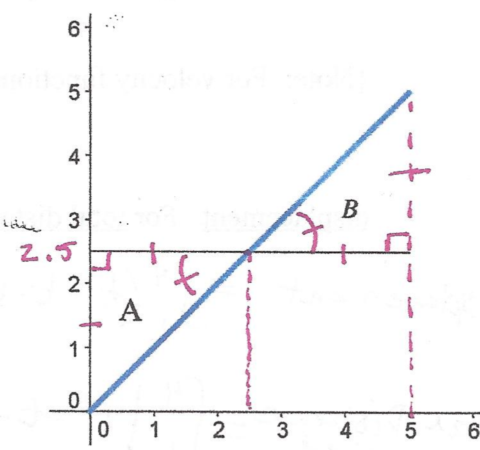
Let's look at this graphically:

Notice the horizontal line at the level of the **average value** on the graph.

7. Show or explain how you know that the area in region *A* is the same as the area in region *B*.

   $\triangle A \cong \triangle B$ by ASA





Playing with blocks:

8. Build a model of $y = x^2$ on [0, 3] with the blocks.
   - What is the average of the data set: 0, 1, 4, 9?
   - Rearrange the blocks into a **rectangle** that has the same base length as your original model.
   - What is the height of this rectangle?

   $\frac{0+1+4+9}{4} = \frac{14}{4} = 3.5$

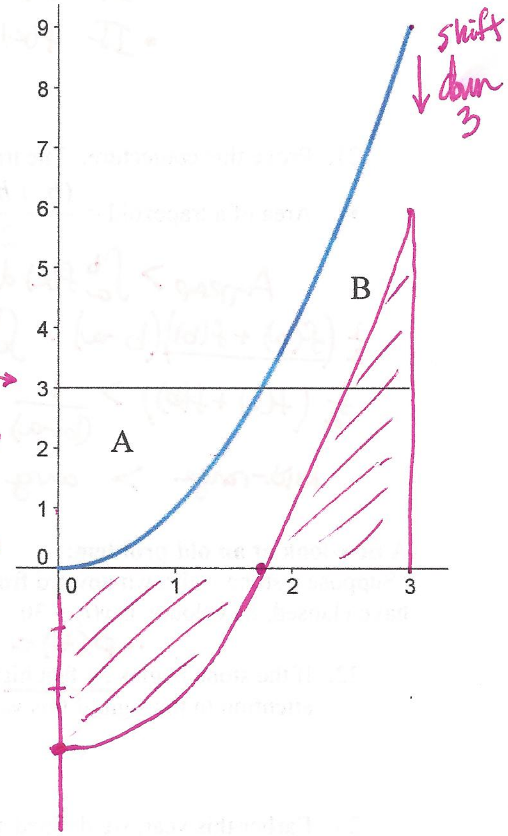
Calculate both the mid-range value and the average value for $f(x) = x^2$. You will have to "think backwards" for the anti-derivative function you need for the average value computation.

9. Mid-range value of $f(x) = x^2$ over $[0, 3]$ = $\frac{9+0}{2} = 4.5$

10. Average value of $f(x) = x^2$ over $[0, 3]$ = $\frac{1}{3-0}\int_0^3 x^2\,dx$

Use definition & calculator

$\frac{1}{3}(9) = 3$



Let's look at this graphically:

11. Draw a horizontal line at the level of the average value on the graph of f over the interval $[0, 3]$. $y = 3$ $\rightarrow$ oops already drawn

12. Use calculus to show that the area of region A is the same as the area of region B.

$x^2 = 3$
$x = \pm\sqrt{3}$
$x = +\sqrt{3}$

$A = \int_0^{\sqrt{3}} (3)\,dx - \int_0^{\sqrt{3}} (x^2)\,dx = \int_0^{\sqrt{3}} (3 - x^2)\,dx$
$= 5.1961 - 1.7320 \qquad \approx 3.646$

$B = \int_{\sqrt{3}}^{3} (x^2)\,dx - \int_{\sqrt{3}}^{3} (3)\,dx = \int_{\sqrt{3}}^{3} (x^2 - 3)\,dx$
$= 7.2679 - 3.8038 \qquad \approx 3.646$

13. A simpler method is to show that

$$\int_0^3 (f(x) - \text{average value of } f)\,dx = 0.$$

- Try this. $\int_0^3 (x^2 - 3)\,dx = 0$ ✓

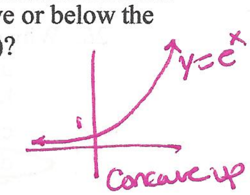
- Why does this method work? Hint: What does the graph of $f(x) -$ average value of f look like?

f(x) - avg value shifts the graph down 3 units so that region A is now below the x-axis & will be negative while region B remains above the x-axis & will be positive. Showing that $\int_0^3 (x^2 - 3)\,dx = 0$ proves the area of region A & region B are equal.

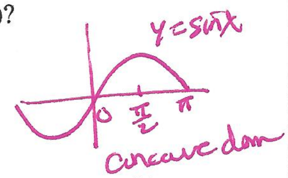
Average vs. mid-range

In the problems in this lab you noticed that the mid-range value was likely to be different than the average value of the function. In this question, we will explore whether there is anything about the shape of the graphs of the functions that could account for this difference.

The graph of $f(x) = e^x$ is concave up on [0, 1]. You will have to "think backwards" for the anti-derivative function you need for the average value computation.

14. Mid-range value of $f(x) = e^x$ over $[0, 1]$ =	15. Average value of $f(x) = e^x$ over $[0, 1]$ =	16. Is the mid-range above or below the average value of $f(x)$?
$\frac{e^1 + e^0}{2} = \frac{e+1}{2}$ ≈ 1.859	$\frac{1}{(1-0)}\int_0^1 e^x\,dx = e^1 - 1$ ≈ 1.718	mid > avg. 

The graph of $f(x) = \sin x$ is concave down on $[0, \frac{\pi}{2}]$.

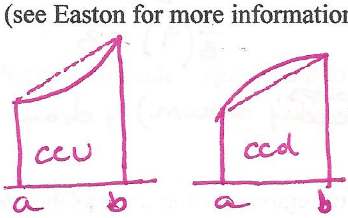
17. Mid-range value of $f(x) = \sin x$ over $[0, \frac{\pi}{2}]$ =	18. Average value of $f(x) = \sin x$ over $[0, \frac{\pi}{2}]$ =	19. Is the mid-range above or below the average value of $f(x)$?
$\frac{\sin\frac{\pi}{2} - \sin 0}{2} = \frac{1}{2}$ 0.50	$\frac{1}{(\frac{\pi}{2} - 0)}\int_0^{\pi/2} \sin x\,dx = \frac{2}{\pi}$ ≈ 0.636	Mid < avg 

20. State a conjecture regarding concavity, mid-range, and average value of a function for a continuous function over the interval $[a, b]$.
- If $f(x)$ is concave up then mid-range > average value.
- If $f(x)$ is concave down then mid-range < average value.

21. Prove this conjecture. The trapezoid rule (see Easton for more information) will help you get started.

- Area of a trapezoid = $\dfrac{(b_1 + b_2)\cdot h}{2}$.

$$A_{TRAP} > \int_a^b f(x)\,dx$$
$$\tfrac{1}{2}(f(a)+f(b))(b-a) > \int_a^b f(x)\,dx$$
$$\tfrac{1}{2}(f(a)+f(b)) > \frac{1}{(b-a)}\int_a^b f(x)\,dx$$
Mid-range > avg value



$$A_{TRAP} < \int_a^b f(x)\,dx$$
$$\tfrac{1}{2}(f(a)+f(b))(b-a) < \int_a^b f(x)\,dx$$
$$\tfrac{1}{2}(f(a)+f(b)) < \frac{1}{(b-a)}\int_a^b f(x)\,dx$$
mid-range < avg value

A new look at an old problem:

"Suppose a stone is thrown upward from a 54 foot high platform with an initial velocity of 30 feet per second. After t seconds have elapsed, its velocity is $v(t) = 30 - 32t$. It lands on the ground 3 seconds after it was thrown."

$\therefore p(t) = 54 + 30t - 16t^2$

22. If the stone begins 54 feet high and lands on the ground, what is the *change in position* of the stone? Please pay attention to the sign of this value.

$0 - 54 = -54$ ft.

23. Earlier this year, we defined average velocity to be: $\dfrac{\textit{change in position}}{\textit{change in time}}$.

$$\frac{p(3)-p(0)}{3-0} = \frac{0-54}{3-0} = -18\ \frac{\text{ft}}{\text{sec}}$$

- Calculate the average velocity of the stone on $[0, 3]$ using this definition.

24. Now, use the new definition for average value of a function to determine the average value of the velocity function on $[0, 3]$.

$$\frac{1}{(3-0)}\int_0^3 v(t)\,dt = \frac{1}{3}\int_0^3 (30-32t)\,dt$$
$$= \frac{1}{3}(-54) = -18\ \frac{\text{ft}}{\text{sec}}$$

25. Use the Fundamental Theorem of Calculus to explain the equality of your answers in parts a and b.

$$\frac{1}{3}\int_0^3 v(t)\,dt = \frac{p(3)-p(0)}{3}$$

26. Why did Ms. Haller have you play with blocks?
   a. She knew you would like it.
   b. To help you visually understand "averaging".
   c. To make you think about the difference between discrete and continuous functions.
   d. All of the above.

Definite Integral as an Average

Think of finding the average of temperatures during the day in your room at n equally spaced times, $t_1, t_2, \ldots\ldots, t_n$. How do you compute the average of the temperatures algebraically?

$$\text{Average temperature } \approx \frac{f(t_1)+f(t_2)+f(t_3)+\ldots\ldots+f(t_n)}{n}$$

The larger we make n, the better the approximation. We can rewrite this expression as a Riemann sum over the interval $0 \le t \le 30$ if we use the fact that $\Delta t = \frac{30}{n}$, so $n = \frac{30}{\Delta t}$:

$$\text{Average temperature } \approx \frac{f(t_1)+f(t_2)+f(t_3)+\ldots\ldots+f(t_n)}{\left(\frac{30}{\Delta t}\right)}$$

$$= \frac{f(t_1)\Delta t+f(t_2)\Delta t+f(t_3)\Delta t+\ldots\ldots+f(t_n)\Delta t}{30}$$

$$= \frac{1}{30}\sum_{i=1}^{n} f(t_i)\Delta t$$

As $n \to \infty$, the Riemann sum tends towards an integral, and 1/30 of the sum also approximates the average temperature better.

$$\text{Average temperature} = \lim_{n\to\infty} \frac{1}{30}\sum_{i=1}^{n} f(t_i)\Delta t = \frac{1}{30}\int_0^{30} f(t)\,dt$$

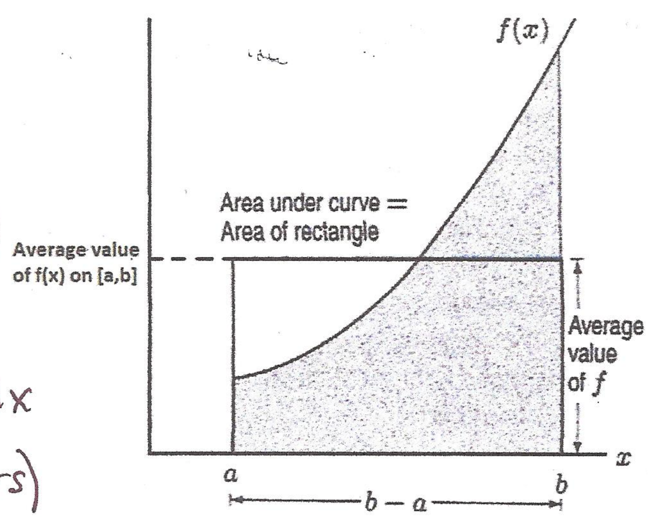
> *Average Value of* $f(x)$
>
> *on interval from* $x = a$ *to* $x = b$
>
> $$Avg\ Value = \frac{1}{b-a}\int_a^b f(x)\,dx$$

How to Visualize the Average on a Graph:

$\boxed{\text{UNITS}}$

$$(\text{Average value of } f)\cdot(b-a) = \int_a^b f(x)\,dx$$

$$\left(\frac{\text{meters}}{\text{hour}}\right) \cdot (\text{hours}) = (\text{meters})$$



$$\text{Average Value} = \frac{1}{(b-a)}\int_a^b f(x)\,dx$$

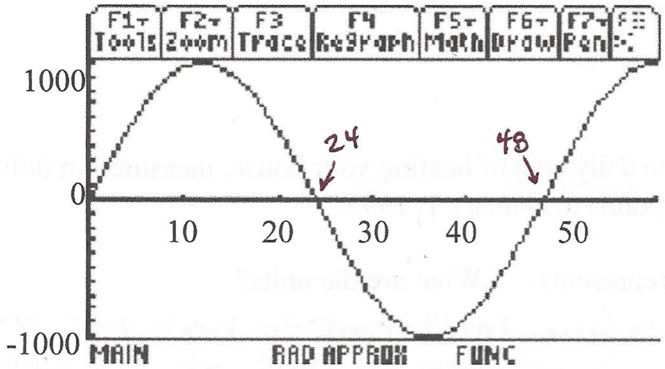
$$\frac{\text{meters}}{\text{hour}} = \left(\frac{1}{\text{hours}}\right)\cdot(\text{meters})$$

KEY

Example 4: An oil spill and its subsequent clean up was monitored. It was determined that the rate of change of the area affected by the spill could be modeled by:

$$R(t) = 1000\sin\left(\frac{\pi}{24}t\right) \quad \frac{ft^2}{hr}$$

where R is measure in ft^2/hr and time, t, is measured in hours. The graph below is of the function $R(t)$.



SOLVE FOR ZEROS

$\sin\left(\frac{\pi}{24}t\right) = 0$

$\frac{\pi}{24}t \in \{0, \pi\} + 2\pi k$

$t \in \{0, 24\} + 48k$

$k \in \mathbb{Z}$

a) Interpret $\int_0^{10} R(t)dt$. $= 5662.205\ ft^2$

Total change in the area of the spill in the first 10 hours is $5662.205\ ft^2$.

b) When will the oil spill be at a maximum?

At time $t = 24$ hours the spill will be at a maximum because $R(t)$ changes signs from $+$ to $-$.

c) When was the oil spill clean up making progress?

At time $t = 48$ hours the spill clean up is making progress because the area of the spill is zero. $\int_0^{48} R(t)\,dt = 0$ -2.1225×10^{-10}

d) Find the average rate at which the oil is spilling out over the time period 0 to 12 hours.

$\int_0^{12} R(t)\,dt = 7639.437268\ ft^2$

Average Rate $= \frac{1}{12}\int_0^{12} R(t)\,dt = 636.6197724\ \frac{ft^2}{hr}$

$= 636.619\ ft^2/hr$
$= 636.620$

Example 5. (Calc) A news broadcast in early 1993 said the average American's annual income is changing at a rate of $r(t) = 40(1.002)^t$ dollars per month, where t is in months from January 1, 1993. How much did the average American's income change during 1993?

Jan 1 1993 $= 0$
Dec 31 1993 $= 12$

Total Change in Annual Income ($) $= \int_0^{12} 40(1.002)^t\,dt = \$485.80$ $\left(\frac{\$}{mo.}\right)(mo.)$

The average American's income in 1993 increased by \$485.80 between Jan 1 & Dec 31, 1993.

1980 1990 2000
0 10 20

KEY p.15

Example 6. (Calc) If the population in Mexico is modeled by the function $P(t) = 67.38(1.026)^t$ where P is in millions and t is in years since 1980, predict the average population of Mexico between the years of 2000 and 2020. What are the units? Show set up.

$$\frac{1}{(40-20)}\int_{20}^{40} 67.38(1.026)^t\,dt = \frac{1}{20}\int_{20}^{40} P(t)\,dt = \frac{1}{20}(2942.6607)$$

≈ 147.133 million people

$\frac{1}{(\text{years})} \cdot (\text{millions})(\text{years}) = \text{millions}$

On average, 147.133 million people live in mexico between year 2000 & 2020.

Example 7. Suppose C(t) represents the daily cost of heating your house, measured in dollars per day, where t is time in days and t = 0 corresponds to January 1, 1997.

a. Explain what the following represents. What are the units?

$\int_0^{90} C(t)dt$ represents the total cost to heat your house over the first 90 days of 1997 where cost is in ~~\$~~ dollars.

Units $\left(\frac{\text{cost}}{\text{day}}\right)(\text{day}) = \text{Cost } \$$

b. Explain what the following represents. What are the units?

$\frac{1}{90-0}\int_0^{90} C(t)dt$ represents the average cost in dollars/day to heat your house over the first 90 days of 1997.

UNITS $\frac{1}{(\text{day})}\left(\frac{\text{cost}}{\text{day}}\right)(\text{day}) = \frac{\text{cost}}{\text{day}}$

Average daily cost to heat your house in first 90 days of 1997.

PRACTICE: Interpret the Meanings of Definite Integral Expressions

1) $C(t)$ is the amount of coffee (in ounces) in a cup at time t, in minutes: Units: $\left(\frac{1}{\text{min}}\right)(\text{ounces})(\text{min}) = \text{ounces}$

$\frac{1}{6-0}\int_0^6 C(t)\,dt = 10.1$ The average amount of coffee in the cup in the interval 0 to 6 minutes is 10.1 ounces.

2) $W(t)$ is the temperature of water (in °F) in a tub at time t, in minutes: Units: $\left(\frac{1}{\text{min}}\right)(°F)(\text{min}) = °F$

$\frac{1}{20-0}\int_0^{20} W(t)\,dt = 60.79$ The average temperature of the water on the interval 0 & 20 minutes is 60.79°F.

3) Johanna's velocity $v(t)$ is measured in meters per minute. Interpret: Units $\left(\frac{\text{meters}}{\text{min}}\right)(\text{min}) = \text{meters}.$

a) $\int_0^{40} |v(t)|\,dt = 7600$

Total distance traveled on interval from 0 to 40 mins is 7600 meters.

b) $\int_0^{40} v(t)\,dt = 5840$

Displacement: Johanna is 5840 meters from her starting position after 0–40 minutes.

4) If $f(t)$ is the amount of cola consumed in the U.S. in billions of gallons per ear and $t = 0$ corresponds to the year 1990, interpret the meaning of

a) $\int_0^{16} f(t)\,dt = 9$

The total change in the amount of cola consumed in the U.S. from 1990 to 2006 is an increase of 9 billion gallons.

b) $\frac{1}{16-0}\int_0^{16} f(t)\,dt = 0.5625$

The average amount of cola consumed in the U.S. from 1990 to 2006 is 0.5625 billions of gallons/year.

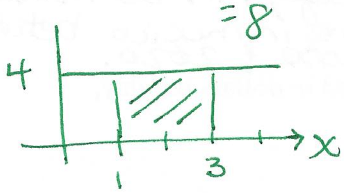
$\boxed{\text{Units}}$: $\left(\frac{\text{billions of gallons}}{\text{year}}\right)(\text{year}) = \text{billions of gallons.}$ $\boxed{\text{Units}}$ $\frac{1}{(\text{year})}\left(\frac{\text{bill of gal}}{\text{year}}\right)(\text{year}) = \frac{\text{bill of gal}}{\text{year}}$

Key p.16

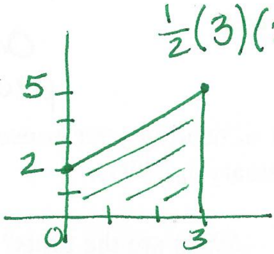
[Practice Problems: The Definite Integral, Area]

Example 8: Sketch the region corresponding to each definite integral. Then evaluate each integral using a geometric formula.

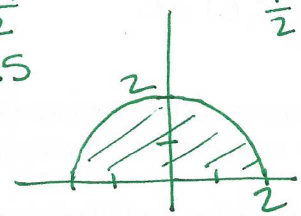
a) $\int_1^3 4dx = 4(3-1)$
$= 4(2)$
$= 8$



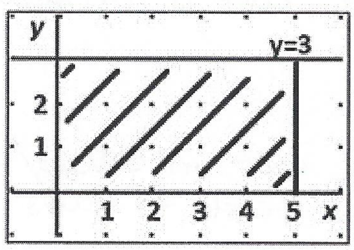
b) $\int_0^3 (x+2)dx =$ $\frac{1}{2}(3)(2+5) = \frac{21}{2}$
$= 10.5$



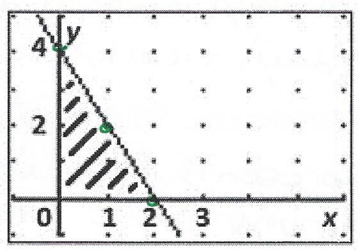
c) $\int_{-2}^{2} \sqrt{4-x^2}\,dx =$ semicircle $r=2$
$\frac{1}{2}(\pi)(2)^2 = 2\pi$



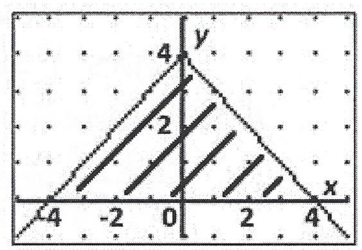
Example 9: Set up a definite integral that yields the area of the region then, evaluate the integral.

a) 

$\int_0^5 3\,dx$
$= 3(5-0)$
$= 15$

b) 

$\int_0^2 (4-2x)\,dx$
$\frac{1}{2}(4)(2)$
$= 4$

c) 

$\int_{-4}^{4} -|x|+4\,dx$
$\int_{-4}^{0} (x+4)\,dx + \int_0^4 (-x+4)\,dx$
$\frac{1}{2}(8)(4) = 16$

Area above x-axis is positive, Area below x-axis is negative.

Example 10: For the graph $f(x)$ given to the right find the following geometrically.

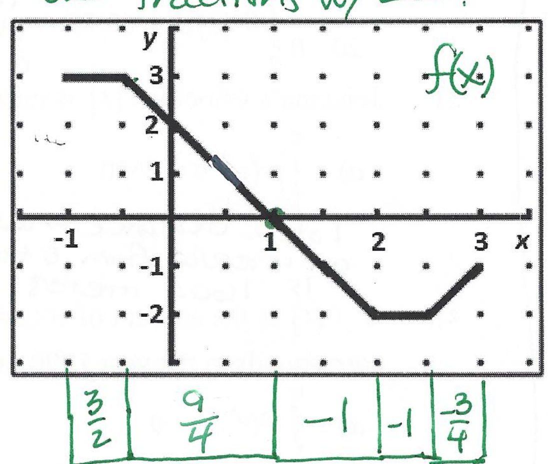
NOTE x-scale is ½
use fractions w/ LCD.

a. $\int_{-1}^{3} f(x)dx =$ 1

$\frac{6}{4} + \frac{9}{4} - \frac{4}{4} - \frac{4}{4} - \frac{3}{4}$
$= \frac{4}{4} = 1$

b. $\int_{0}^{3} f(x)dx =$ $-\frac{7}{4}$

$2 - 2 - 1 - \frac{3}{4} = -\frac{7}{4}$



$\frac{3}{2}$ | $\frac{9}{4}$ | -1 | -1 | $-\frac{3}{4}$
$\frac{6}{4}$ $\frac{9}{4}$ $-\frac{4}{4}$ $-\frac{4}{4}$ $-\frac{3}{4}$

c. average value over [-1, 3]

$\frac{1}{(3+1)}\int_{-1}^{3} f(x)\,dx = \frac{1}{4}(1) = \frac{1}{4}$

d. $f'(1) =$ -2

Slope of f(x) graph at x=1

e. $f''(0) =$ 0

$f'(0) = -1$
$f''(0) = 0$

f. average value over [0, 3]

$\frac{1}{(3-0)}\int_0^3 f(x)\,dx$
$\frac{1}{3}\left(\frac{-7}{4}\right) = \frac{-7}{12}$