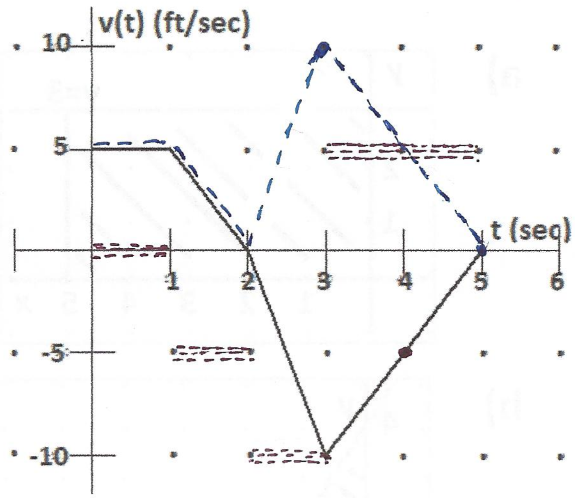
KEY

AB CALCULUS: §5.3 **Position vs Distance**
Velocity vs. Speed
Acceleration

a(t) ------ graph
speed = |v(t)| ---- graph.



All the questions which follow refer to the graph at the right.

1. When is the particle moving at a constant speed?
   $t \in (0,1)$ seconds b/c $v(t) = 5$ ft/sec.
2. When is the particle moving to the right? Why?
   $t \in (0,2)$ seconds b/c $v(t) > 0$.
3. When is the particle moving to the left? Why?
   $t \in (2,5)$ seconds b/c $v(t) < 0$.

Draw a(t) graph.

*4. When is the particle speeding up? Why?
   $t \in (2,3)$ seconds b/c $v(t) < 0$ & $a(t) < 0$.
5. When is the particle slowing down? Why?
   $t \in (1,2), (3,5)$ b/c $v(t)$ & $a(t)$ have opposite signs.
6. When is the velocity increasing? Why?
   $t \in (3,5)$ seconds b/c $v'(t) > 0$ or $a(t) > 0$.
7. When is the velocity decreasing? Why?
   $t \in (1,3)$ seconds b/c $v'(t) < 0$ or $a(t) < 0$.
8. Are your answers to questions 6 & 7 the same as your answers to 4 & 5 respectively? Explain.
   No #4 & 6 do not have the same answer.
   No #5 & 7 do not have the same answer.
   Velocity is increasing when it's slope is positive, decreasing when its slope is negative.
   Particle is speeding up when $v(t)$ & $a(t)$ have the same sign, slowing down when they have opposite signs.
9. How fast is the particle moving at time t = 4 and in what direction?
   $v(4) = -5$ ft/sec means the particle is moving 5ft/sec to the left.
10. What is the particle's velocity at time t = 4? $v(4) = -5$ ft/sec.
11. When does the particle change direction? The particle changes direction at time $t = 2$ sec b/c $v(t)$ changes signs (+) to (−). The particle changes from increasing (moving right) to decreasing (moving left).
12. How far does the particle move during the first second and which way?
    The particle moves 5ft to the right in the initial second. b/c $\left(\frac{5\text{ft}}{\text{sec}}\right)(1\text{sec}) = 5\text{ft}$.
13. How far to the right does the particle go and when does it arrive there?
    The particle moves $\frac{1}{2}(5)(1+2) = \frac{15}{2} = 7.5$ ft to the right in the first 2 seconds.
    [The particle moves $\frac{1}{2}(3)(-10) = -15$ ft or 15 ft left on $t \in (2,5]$ seconds.]
14. What is the total distance traveled by the particle?
    $7.5 + 15 = 22.5$ ft = total distance traveled. $\int_0^5 |v(t)|\,dt = 22.5$ ft.
15. What is the particle's position after 5 secs?
    Displacement $= 7.5 - 15 = -7.5$ ft
    The particle is 7.5 ft to the left
    $\int_0^5 v(t)\,dt = -7.5$ ft.
16. Are your answers to questions 14 & 15 the same? Explain.
    No. #14 finds total distance = 22.5 ft
    #15 finds the displacement = −7.5 ft

KEY

17. What is the average velocity of the particle over the 5 second time interval?

$\frac{1}{(5-0)}\int_0^5 v(t)\,dt = \frac{1}{5}(-7.5) = -1.5 \frac{ft}{sec}$

$\left(\frac{1}{sec}\right)\cdot\left(\frac{ft}{sec}\right)(sec) = ft/sec$

18. What is the average velocity over the time interval 2< t <3?

$\frac{1}{3-2}\int_2^3 v(t)\,dt = 1\left(\frac{1}{2}\right)(1)(10) = 5 \frac{ft}{sec}$

$\left(\frac{1}{sec}\right)\cdot\left(\frac{ft}{sec}\right)(sec)$

19. What is the average speed over the time interval of #18?

*Explain.

Speed $= |v(t)|$ over $2<t<3$

$= 5 ft/sec$

$\frac{1}{1}\int_2^3 |v(t)|\,dt$

20. What is the particle's acceleration over the 1st second? $0\ ft/sec^2$ What does your answer mean?

$a(t) = v'(t) = 0\ ft/sec^2$

The particle is moving at a constant velocity 5ft/sec & is neither speeding up or slowing down.

21. Over what time interval(s) does the particle have positive acceleration? Why?

$a(t) > 0$ on $(3,5)$ seconds.

& $v'(t) > 0$

22. Over what time interval(s) does it have negative acceleration? Why?

$a(t) < 0$ on $(1,3)$ seconds.

& $v'(t) < 0$

23. When is the acceleration undefined on the interval 0 < t < 5?

$t = 1, 2, 3$ b/c $v'(t^-) \neq v'(t^+)$ at these t-values.

∴ sharp points ∴ $v(t)$ is not differentiable.

24. State an interval of time when the particle has positive acceleration but is slowing down. Explain.

$a(t) > 0$ & $v(t) < 0$ on $t \in (3,5)$.

25. State an interval when the acceleration is negative but the particle is speeding up. Explain.

$a(t) < 0$ & $v(t) < 0$ on $t \in (2,3)$.

26. What is the particle's acceleration over the time interval (3, 4)?

$a(t)$ on $(3,4) = \frac{-5 - -10}{4-3} = \frac{5\ ft/sec}{1\ sec} = 5\ ft/sec^2$

27. What is the particle's average acceleration over the entire 5second time interval?

wording is just like #17

$\frac{1}{(5-0)}\int_0^5 a(t)\,dt = \frac{1}{5}(-5 + -10 + 10) = \frac{1}{5}(-5) = -1\ ft/sec^2$

$\left(\frac{1}{sec}\right)\left(\frac{ft}{sec^2}\right)(sec) = \frac{ft}{sec^2}$

28. Over what time interval is the particle's velocity decreasing at a rate of 5 feet/sec every second?

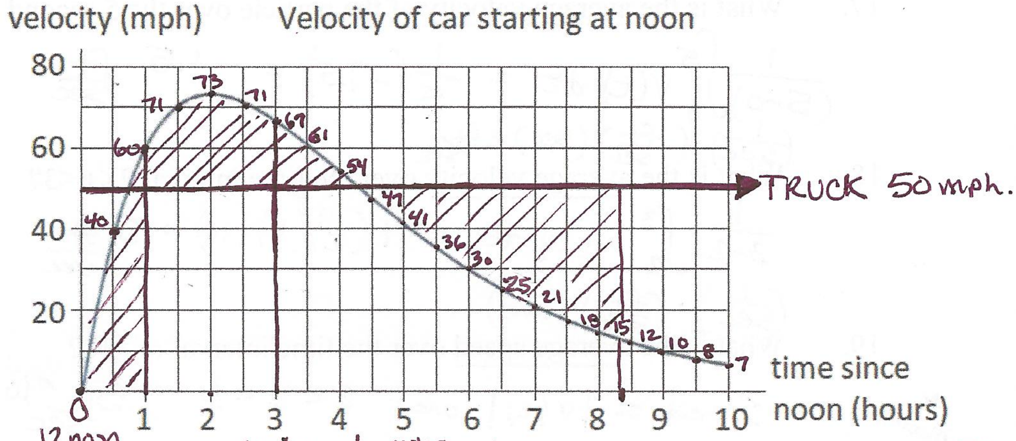
$a(t) = v'(t) = -5\ ft/sec^2$ on $t \in (1,2)$.

KEY

AB Calculus §5.3

The Car and Truck Problem

A car starts at noon and travels with the velocity shown in the figure. A truck starts at 1 pm from the same place and travel at a constant velocity of 50 mph.



12noon (CAR)

t	miles car	miles truck	Δmiles	TRAP RULE ESTIMATES
0	0	0	0	
1	35	0	35	$\frac{1}{2}(\frac{1}{2})(0+2(40)+60)$
2	67.75	50	17.75	$\frac{1}{4}(0+2(40+60+71)+73)$
3	174.25	100	74.25	$\frac{1}{4}(0+2(40+60+71+73+71)+67)$
4	235	150	85	
* 5	282.25	200	82.25	$\frac{1}{4}(0+2(40+60+71+73+71+67+61)+54)$
6	318	250	68	
7	343.25	300	43.25	
8	361.25	350	11.25	
9	373.5	400	−26.5	
10	381.75	450	−68.25	

1. How far away is the car when the truck starts?

$\frac{1}{2}(60)(1) = 30^{+}$ miles

better $(\frac{1}{2})(\frac{1}{2})(0+40+40+60) = 35$ miles

2. a. How far has the car gone at 3:00?

$(\frac{1}{2})(\frac{1}{2})(0+2(40+60+71+73+71)+67)$

$\approx 174.25 \therefore \int_0^3 v(t)\,dt \approx 174.25$ miles.

b. What was the car's velocity at 3:00?

$v(3) \approx 67$ mph.

read from graph.

3. What was the car's average velocity over the first three hours of its trip?

$$\frac{1}{(3-0)}\int_0^3 v(t)\,dt = \frac{1}{3}(174.25) \approx 58 \text{ mph.}$$

$$\frac{1}{(hr)}\left(\frac{miles}{hr}\right)\cdot hr$$

4. Graph the truck's velocity function. ✓

5. At what point do the two velocity functions intersect? What is the relevance of this point besides the obvious fact that they are traveling at the same speed then?

Approximately 4 hours 15 minutes @ $t = 4.25$ hours

* Some time between $t = 4$ hr & $t = 5$ hr the distance between the car & truck is the greatest.

6. During the period of time when the car is ahead of the truck:

$\to t \in (0, 8)$ ... between $t = 8$ & 9 the truck over takes the car.

a. When is the distance between them the greatest?

between $t \in (4,5) \to$ approximately 4hr 15min

b. Approximate this distance between them

$t \in (0,4)$ 235

$t \in (4, 4.25)$ 13

car distance 248

truck distance $(50)(4.25) = 212.50$

Distance between car & truck = 35.5 miles.

7. a. When does the truck overtake the car?

Sometime between $t \in (8,9)$ hrs

approximately 8hrs 20 minutes.

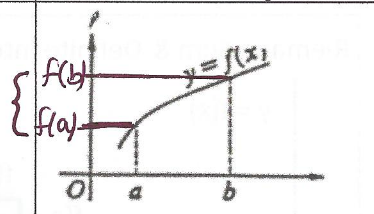
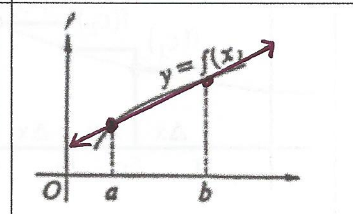
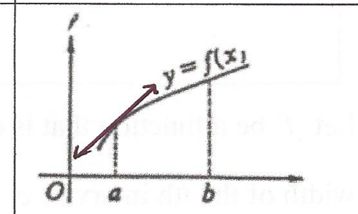
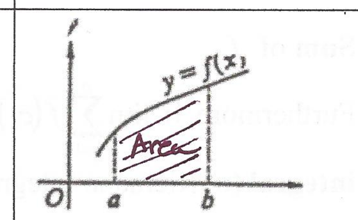
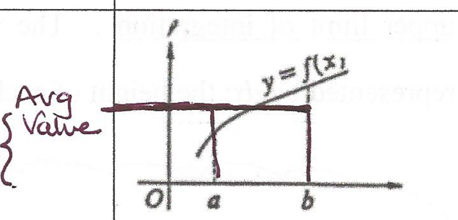
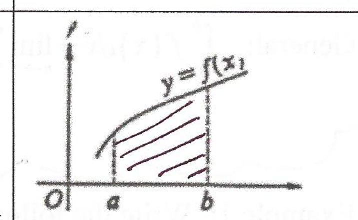
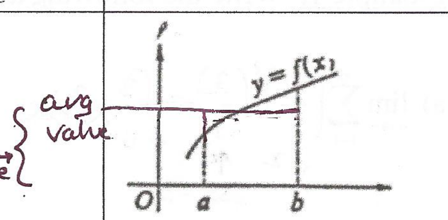
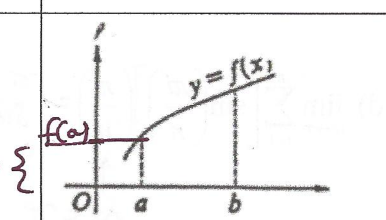
b. How far have both traveled then?

hrs	car	truck
8	361.25	350
9	373.5 miles	400 miles

KEY

AP Calculus AB Geometric Understanding Fundamental Theorem of Calculus

Complete the table below. Let $f = F'$ ∴ f is a RATE

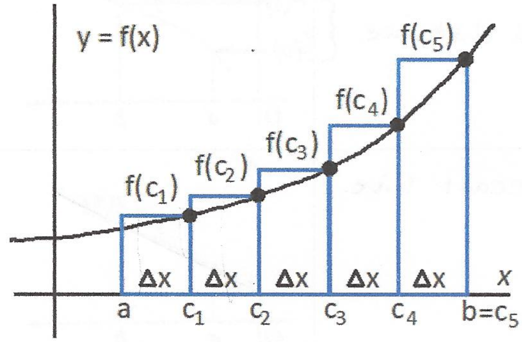
Expression	What It's Called	Is it a Length, Area, or Slope on f?	Sketch on f
$f(b)-f(a)$	Change in f ~~rate~~.	length – the vertical distance {	
$\dfrac{f(b)-f(a)}{b-a}$	• slope of secant • F'' estimate on [a,b]. • average rate of change of f.	slope of secant line	
$f'(a)$	• slope of tangent line at x=a. • F'' at x=a • instantaneous rate of change	slope of tangent line.	
$\int_a^b f(t)dt$	Total Change in F(x) btwn x=a & x=b. F(b) – F(a)	area under f(x) and above x-axis on interval from x=a to x=b.	
$\dfrac{1}{b-a}\int_a^b f(t)dt$	Average value of f(x) $\frac{1}{(b-a)}(F(b)-F(a))$	Length vertical distance → {	
$F(b)-F(a)$	Total change on F(x) btwn x=a & x=b.	area between f(x) and x-axis from x=a to x=b.	
$\dfrac{F(b)-F(a)}{b-a}$	Average value of f(x) on (a,b).	Length vertical distance → {	
$f = F'$ $F'(a)$ $= f(a)$	y-value at x=a	length vertical distance → {	

KEY

§5.4 Properties of the Definite Integral

Let's use the following graph to develop an understanding for notation of the Riemann Sums.

Riemann Sum & Definite Integral



$\int_a^b f(x)dx$ can be approximated by:

$$\text{area} = f(c_1)\Delta x + f(c_2)\Delta x + f(c_3)\Delta x + f(c_4)\Delta x + f(c_5)\Delta x$$
$$= [f(c_1) + f(c_2) + f(c_3) + f(c_4) + f(c_5)]\Delta x$$
$$= \sum_{i=1}^{5} f(c_i)\Delta x$$

& if we use ***n*** rectangles instead of 5 we write: $\sum_{i=1}^{n} f(c_i)\Delta x_i$

$n \to \infty$

What will happen if we let Δx get smaller and smaller?
The width of each rectangle will become smaller and smaller.
The number of rectangles will increase.

We can represent this by using a limit: $\int_a^b f(x)dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(c_i)\Delta x_i$

Let f be a function that is defined on the closed interval $[a,b]$. If Δ is a partition of $[a,b]$ and Δx_i is the width of the ith interval, c_i is any point in the subinterval, then the sum $\sum_{i=1}^{n} f(c_i)\Delta x_i$ is called a **Riemann Sum** of f.

Furthermore, if $\lim_{n\to\infty}\sum_{i=1}^{n} f(c_i)\Delta x_i$ exists, we say f is **integrable** on $[a,b]$ and $\int_a^b f(x)dx$ is called the **definite integral** (or Riemann Integral) of f from a to b. a is called the lower limit of integration and b is called the upper limit of integration. The width of each rectangle is $\Delta x_i = \dfrac{b-a}{n}$ and in the definite integral is represented by dx; the height of each rectangle is $f(c_i)$ and in the definite integral is the function.

General: $\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(c_i)\Delta x_i$

Examples:

$$\int_0^5 \sqrt[3]{x+7}\,dx = \lim_{n\to\infty}\sum_{i=1}^{n}\left(\sqrt[3]{\frac{5i}{n}+7}\right)\left(\frac{5}{n}\right)$$

Differences

$$\int_1^5 \sqrt[3]{x+7}\,dx = \lim_{n\to\infty}\sum_{i=1}^{n}\left(\sqrt[3]{\left(1+\frac{4i}{n}\right)+7}\right)\left(\frac{4}{n}\right)$$

Example 1: Write the following as definite integrals:

a) $\lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{3i}{n}\right)^2\left(\frac{3}{n}\right) = \int_0^3 x^2\,dx$ $\Delta x = 3$

b) $\lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{2i}{n}\right)^3\left(\frac{2}{n}\right) = \int_0^2 x^3\,dx$ $\Delta x = 2$

c) $\lim_{n\to\infty}\sum_{i=1}^{n}\left[1+\frac{2i}{n}+\left(\frac{2i}{n}\right)^2\right]\left(\frac{2}{n}\right)$ $\int_0^2 (1+x+x^2)\,dx$ $\Delta x = 2$

Compare

d) $\lim_{n\to\infty}\sum_{i=1}^{n}\left[\sin\left(\frac{\pi i}{n}\right)\right]\left(\frac{\pi}{n}\right) = \int_0^{\pi} \sin x\,dx$ $\Delta x = \pi$

e) $\lim_{n\to\infty}\sum_{i=1}^{n}\left[\sqrt{\frac{6i}{n}}\right]\left(\frac{2}{n}\right) = \int_0^2 \sqrt{3x}\,dx$ $\Delta x = 2$

f) $\lim_{n\to\infty}\sum_{i=1}^{n}\left[1+\frac{4i}{n}+\left(\frac{2i}{n}\right)^2\right]\left(\frac{2}{n}\right)$ $\int_0^2 (1+2x+x^2)\,dx$ $\Delta x = 2$

Calculator activity to demo. properties

KEY

ODD $f(x)$ $\int_{-a}^{a} f(x)\,dx = 0$

EVEN $f(x)$ $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$

Properties of Integration:

$\int_a^a f(x)dx = 0$

$\int_a^b c\,dx = (b-a)\cdot c$ (c is any constant)

* $\int_a^b f(x)dx = -\int_b^a f(x)dx$

$\int_a^b k\cdot f(x)dx = k\int_a^b f(x)dx$

$\int_a^b (f(x)+g(x))dx = \int_a^b f(x)dx + \int_a^b g(x)dx$

$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$

Example 2: Suppose $\int_0^1 f(x)dx = 4$ and $\int_0^1 g(x)dx = -2$, find:

a) $\int_0^1 3f(x)dx = 3\int_0^1 f(x)dx$
$= 3(4)$
$= 12$

b) $\int_0^1 [f(x)-g(x)]dx = 4 - -2$
$= 6$

c) $\int_0^1 [3f(x)+2g(x)]dx$
$3\int_0^1 f(x)dx + 2\int_0^1 g(x)dx$
$= 3(4) + 2(-2) = 8$

d) $\int_0^1 [2g(x)-3f(x)]dx$
$2\int_0^1 g(x)dx - 3\int_0^1 f(x)dx$
$= 2(-2) - 3(4)$
$= -16$

e) $\int_1^0 [2f(x)]dx = -2\int_0^1 f(x)dx$
$= -2(4)$
$= -8$

Example 3: Given $\int_a^b f(x)dx = 5$ *and* $\int_a^b g(x)dx = -4$, use the properties to evaluate the following:

a. $\int_b^a f(x)dx$
$= -\int_a^b f(x)$
$= -5$

b. $\int_a^b 3f(x)dx$
$= 3\int_a^b f(x)dx$
$= 3(5)$
$= 15$

c. $\int_a^b 3dx$
$= 3(b-a)$
$= 3b - 3a$

d. $\int_a^b (3+f(x))dx$
$= 3(b-a) + (4)$
$= 3b - 3a + 4$

e. $\int_a^b (f(x)+g(x))dx$
$= \int_a^b f(x)dx + \int_a^b g(x)dx$
$= 5 - 4$
$= 1$

f. $\int_a^b (f(x)-g(x))dx$
$= \int_a^b f(x)dx - \int_a^b g(x)dx$
$= 5 - -4$
$= 9$

g. $\int_a^6 g(x)dx + \int_6^b g(x)dx$
$= \int_a^b g(x)dx$
$= -4$ if $a < 6 < b$

KEY

4. If $\int_2^5 (2f(x)+3)dx = 17$, find $\int_2^5 f(x)dx$

$2\int_2^5 f(x)\,dx + \int_2^5 3\,dx = 17$

$2\int_2^5 f(x)\,dx + 3(5-2) = 17$

$2\int_2^5 f(x)\,dx + 9 = 17$

$2\int_2^5 f(x)\,dx = 8$

$\int_2^5 f(x)\,dx = 4$

5. If $f(x)$ is an even function and you know $\int_0^4 f(x)dx = 6$, then what is the value of $\int_{-4}^4 f(x)dx$?

$\int_0^4 f(x)\,dx = 6 \quad \therefore \int_{-4}^4 f(x)\,dx = 2\int_0^4 f(x)\,dx = 12$

6. If $f(x)$ is an odd function and you know $\int_0^4 f(x)dx = 6$, then what is the value of $\int_{-4}^4 f(x)dx$?

$\int_0^4 f(x) = 6$

$\int_{-4}^4 f(x)\,dx = \int_{-4}^0 f(x)\,dx + \int_0^4 f(x)\,dx$

$= -6 + 6$

$= 0$

7. Given: $\int_0^7 f(x)dx = 5$, $\int_5^7 f(x)dx = 11$, $\int_7^{10} f(x)dx = 12$ evaluate each of the following integrals:

a. $\int_5^{10} f(x)dx$

$= \int_5^7 f(x)\,dx + \int_7^{10} f(x)\,dx$

$= 11 + 12$

$= 23$

b. $\int_4^4 f(x)dx$

$= 0$

c. $\int_7^0 f(x)dx$

$= -\int_0^7 f(x)\,dx$

$= -5$

d. $\int_0^5 f(x)dx$

$= \int_0^7 f(x)\,dx - \int_5^7 f(x)\,dx$

$= 5 - 11$

$= -6$

e. $\int_5^7 4f(x)dx$

$= 4\int_5^7 f(x)\,dx$

$= 4(11)$

$= 44$

f. $\int_{10}^7 \frac{1}{2} f(x)dx$

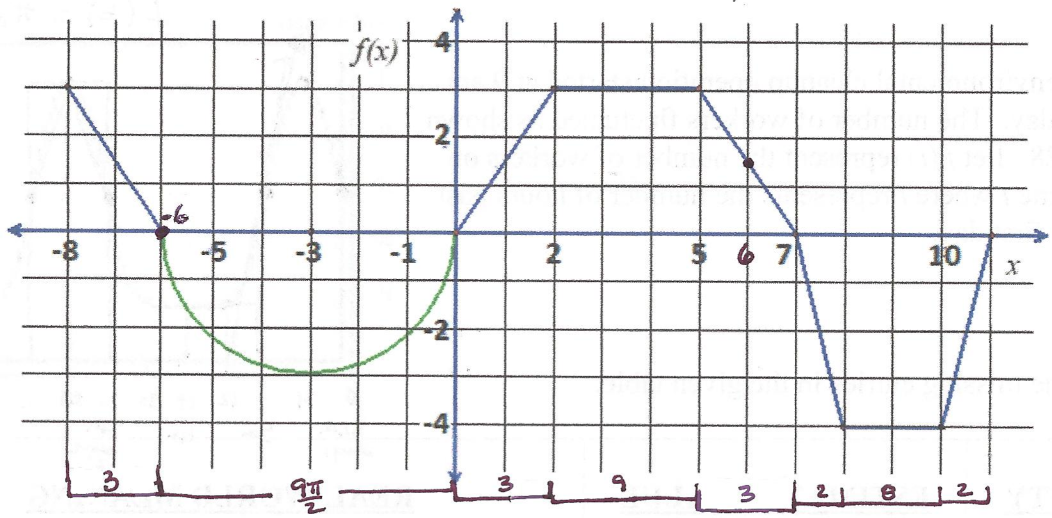
$= -\frac{1}{2}\int_7^{10} f(x)\,dx$

$= -\frac{1}{2}(12)$

$= -6$

WS: **§5.4 Properties of the Definite Integral** KEY

Part I: Given the graph of $f(x)$ on the domain $[-8,11]$.



1. $\int_0^7 f(x)dx = 15$

2. $\int_0^{11} f(x)dx$
$= 15 - 12$
$= 3$

3. $\int_{-6}^0 f(x)dx$
$= -\frac{9\pi}{2}$

4. $\int_4^4 f(x)dx = 0$

5. $\int_7^0 f(x)dx$
$= -\int_0^7 f(x)dx$
$= -15$

6. $\int_8^4 f(x)dx$
$= -\int_4^8 f(x)\,dx$
$= -(6-2) = -4$

7. $\int_{-8}^{11} f(x)dx$
$= 3 - \frac{9\pi}{2} + 15 - 12$
$= 6 - \frac{9\pi}{2} = \frac{12-9\pi}{2}$

8. Average value in f over $[0, 5]$
$\frac{1}{5-0}\int_0^5 f(x)dx$
$= \frac{1}{5}(12) = \frac{12}{5}$

9. $f'(-6)$ undefined
due to sharp point
$f'(-6^-) = -\frac{3}{2} \neq f'(-6^+) = \infty$ undefined vertical tangent

10. $f''(6) = 0$
$f'(6) = -\frac{3}{2} \therefore f''(6) = 0$

11. $\boxed{\text{True}}$ or False: $\underbrace{\int_2^4 f(x)dx + \int_4^6 f(x)dx}_{} = \int_2^6 f(x)dx$

12. $\boxed{\text{True}}$ or False: $\int_2^0 f(x)dx + \int_0^6 f(x)dx = \int_2^6 f(x)dx$
$-\int_0^2 f(x)dx + \int_0^6 f(x)\,dx = \int_2^6 f(x)dx$

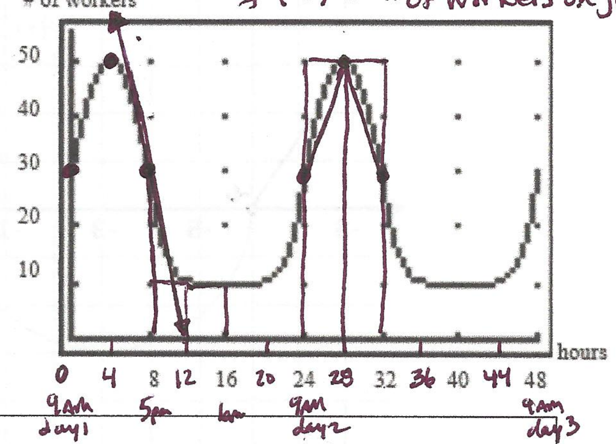
13. Over what values of x is f not differentiable?
$x \in \{-6, 0, 2, 7, 8, 10\}$
b/c $f(x)$ has sharp points
$f'(x^-) \neq f'(x^+)$ at these x-values.

WS: §5.4 Properties of the Definite Integral

Part II. The Worker Problem (Hughes-Hallett, 2nd ed., p. 165, #12 – Revised)

$f(t) = $ # of workers on job

A two-day environmental cleanup operation started at 9 am on the first day. The number of workers fluctuated as shown in figure 3.28. Let $f(t)$ represent the number of workers on the job at time t where t represents the number of hours past 9 am on the first day.



Complete the missing entries in the given table.

QUANTITY	ESTIMATED VALUE (include units)	REAL-WORLD MEANING (as a complete sentence)
1. $f(8)$	30 workers	At 5pm on the first day of cleanup there were 30 workers on the job.
2. $f(8) - f(4)$	50 − 30 = 20 workers	20 workers left the job site between 1 pm & 5pm.
3. $\dfrac{f(4)-f(0)}{4-0}$	$\dfrac{50-30}{4-0} = \dfrac{20}{4} = 5$ 5 workers per hour.	Between 9 AM & 1pm the number of workers increased at a rate of 5 workers per hour.
4. $f'(t) = 0$ *t-values	0 workers/hour rate	* $t \in \{4, 28\}$, $t \in (12,20), (36,44)$ satisfy $f'(t)=0$
5. $f'(8)$	estimate slope of tangent line $\dfrac{0-60}{12-4} = \dfrac{-60}{8} = -7.5 \dfrac{\text{workers}}{\text{hr.}}$	At 5pm, 8 hrs after cleanup began, the number of workers on the job is decreasing at a rate of 15 workers every 2 hours.
6. $\int_{24}^{32} f(t)dt$	(show work) estimate TRAPs $\frac{1}{2}(4)(30+50+50+30)$ ≈ 320 workers	Between 9AM & 5 pm on the 2nd day of cleanup there were approximately 320 workers on the job. (This is an underestimate).
7. $(80)\cdot\int_{24}^{32} f(t)dt$	(workers)(hrs)($/hr) (32−24) (320)(8)($10/hr) $25,600	The cost of the cleanup operation from 9 am to 5 pm on the 2nd day if workers were paid $10/hr. Cost to pay workers
8. $\dfrac{f(16)-f(8)}{16-8}$	$\dfrac{10-30}{16-8} = \dfrac{-20}{8} = \dfrac{-5}{2}$ Labor force declined 5 workers every 2 hours.	The average rate at which the labor force declined from 5 pm to 1 am on the 1st day. average rate of change $= \frac{\Delta y}{\Delta x}$
9. $\dfrac{1}{8}\int_{8}^{16} f(t)\,dt$	$\dfrac{(8)(10)+\frac{1}{2}(4)(20)}{8 \text{hrs}} = \dfrac{120}{8}$ 15 workers/hr.	The average number of workers during the time period 5 pm to 1 am on the 1st day. average value $= \frac{1}{b-a}\int_a^b f(x)dx$