KEY

DAY 84

§5.1 How do we Measure Distance Traveled given Velocity? – Student Notes

EX 1) The table contains velocities of a moving car in ft/sec for time t in seconds:

time (sec)	0	1	2	3	4	5
velocity (ft/sec)	10	15	25	45	30	50

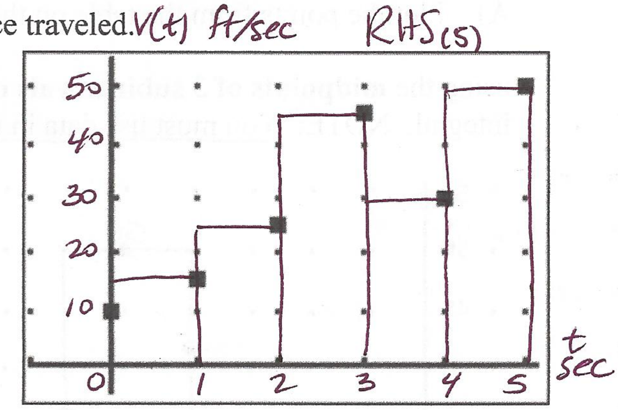
A) Label the x-axis & y-axis for the graph of these points.

- Draw rectangles using the left-hand velocities as the heights of the rectangles.
- What unit is associated with the width & height of each rectangle? W=Δt=1second h=v(t) ft/sec
- What does each rectangle area represent and what is its unit?
  The distance the car travels, in feet, in each 1-second interval.
  (sec)(ft/sec) = ft
- Since each rectangle area represents a DISTANCE, the sum of the rectangle areas may be used to estimate the total distance the car has traveled.

$distance = (\underset{\Delta t}{1})[10 + 15 + 25 + 45 + 30]$

$LHS_5 = 125 \text{ ft}$

We call this a **Left-Hand Riemann Sum** estimate for total distance traveled.



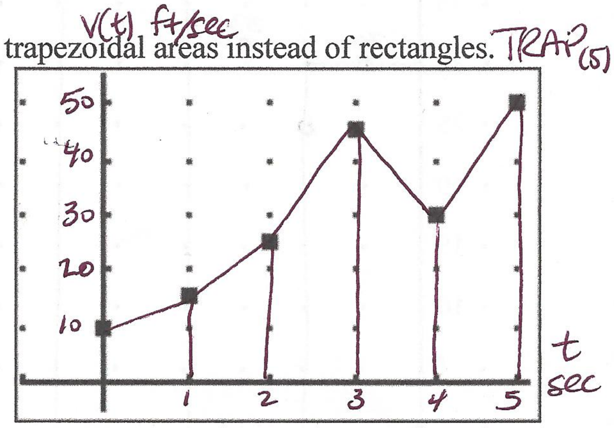
B) Again, label the x-axis & y-axis for the graph. This time draw rectangles using the right-hand velocities as the heights of the rectangles.

$distance = (\underset{\Delta t}{1})[15 + 25 + 45 + 30 + 50]$

$RHS_5 = 165 \text{ ft}$

We call this a **Right-Hand Riemann Sum** estimate for total distance traveled.

C) Another way to estimate the total distance traveled is by using trapezoidal areas instead of rectangles. Again, label the x-axis & y-axis for the graph. This time connecting each height with a segment to create a trapezoid. Find the area of each trapezoid and then the sum of these areas.



Recall the area formula of a trapezoid: $\frac{1}{2}(h)(b_1+b_2)$

$distance = (\frac{1}{2}(1))[10+15 + 15+25 + 25+45 + 45+30 + 30+50]$

$TRAP_5 = 145 \text{ ft}$

D) Find the average of the LHS and the RHS: $\frac{125+165}{2} = 145 \text{ ft}$

- How does this compare to the **Trapezoid Riemann Sum**? equal.
  $\frac{LHS+RHS}{2} = TRAP$
- What is the estimate of the total distance traveled in the first five seconds? 145 ft
  is the best estimate of the three.

Let's look at the notation that represents the exact value for the total distance traveled rather than the Riemann Sum estimates (LHS, RHS and TRAP) for the areas that we found on the previous page.

$\boxed{\int_0^5 v(t)dt}$ We read this as "the definite integral of v(t) with respect to t on the interval from t=0 to t=5."

We interpret this as "the area between the curve v(t) and the t-axis from t=0 to t=5."

Each product $v(t)\cdot dt$ represents the distance in feet the car travels in a given time interval.

$\left(\frac{ft}{sec}\right)(sec) = ft$

If we allow dt, the width of each rectangle, to get infinitesimally small then the number of rectangles will become infinitely large. Sum up all of the areas of these rectangles and we will have an exact value for the distance traveled.

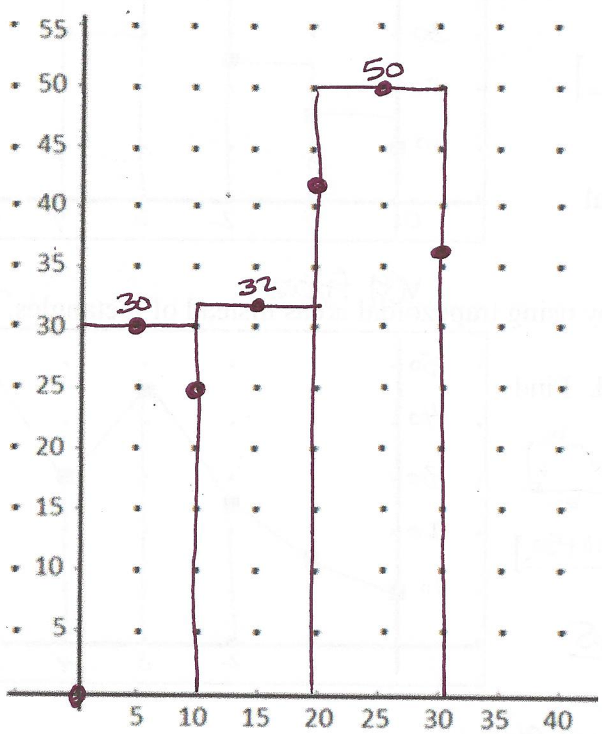
The integral symbol $\int$ means summation. The integral symbol $\int_0^5$ has limits of integration with the lower limit $t=0$ sec and the upper limit $t=5$ sec which represent the left and right bounds of the area we are finding.

EX 2) A polar bear is moving through the water, followed by a kayak of Eskimos, during a 30 minute time interval. A table of speeds of the bear, $v(t)$, is shown for 5 minute intervals of time, t.

MID $\Delta t = 10$ min MID $v(t) = \square$

t (min)	0	5	10	15	20	25	30
$v(t)$ ft/min	0	$\boxed{30}$	25	$\boxed{32}$	42	$\boxed{50}$	36

A) Plot the points from the table on the graph below then approximate $\int_0^{30} v(t)dt$ with a Riemann sum, using the **midpoints of 3 subintervals of equal lengths**. Using correct units, explain the meaning of this integral. NOTE: You must use data in the table. You cannot invent data to be midpoint height values.



Midpoint Riemann Sum:

$\Delta t = 10$ min $v(t)$ hts: 30, 32, 50

$MID_3 = (10)(30+32+50)$

$= 10(112)$

$= 1120$ ft

$\int_0^{30} v(t)\,dt \approx 1120\text{ft} = MID(3)$

B) Find the other 3 Riemann Sum estimates

i) $LHS_6 = 5(0+30+25+32+42+50)$
$= 895$ ft

ii) $RHS_6 = 5(30+25+32+42+50+36)$
$= 1075$ ft

iii) $TRAP_6 = \frac{1}{2}(5)\left(0+2(30+25+32+42+50)+36\right)$
$= 985$ ft.

In this unit we will learn how these estimates compare to each other and to the actual integral sum.

$\frac{LHS+RHS}{2} = TRAP$

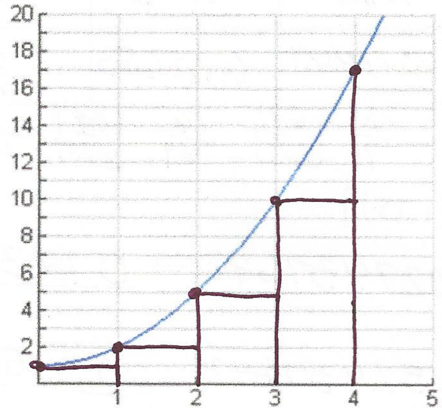
$\frac{895+1075}{2} = 985$ ft

KEY

§5.2 & §7.5 Comparison of Riemann Sum Approximations – Student Notes

For each Riemann Sum: (a) draw rectangles or trapezoids representing each geometric approximation.
(b) Show the calculation to estimate the definite integral.
(c) Is the Riemann Sum approximation an underestimate or overestimate?

Figure 1: $f(x)=x^2+1$



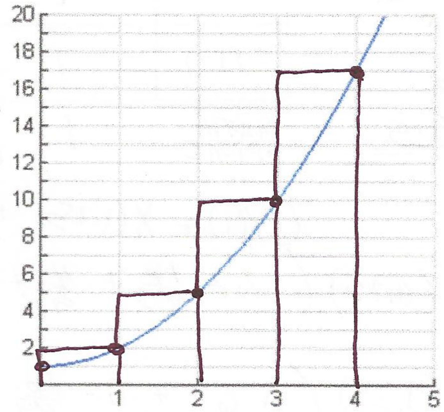
x	y
0	1
1	2
2	5
3	10
4	17

Use **LHS₄** to approximate $\int_0^4 (x^2+1)dx$.

$LHS_4 = (1)(1+2+5+10)$
$= 18$

Under or Over estimate?

Figure 2: $f(x)=x^2+1$

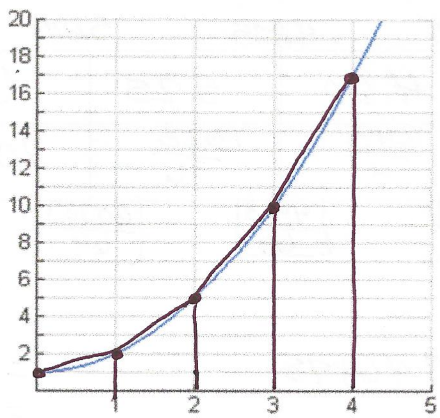


Use **RHS₄** to approximate $\int_0^4 (x^2+1)dx$

$RHS_4 = (1)(2+5+10+17)$
$= 34$

Under or Over estimate?

Figure 3: $f(x)=x^2+1$

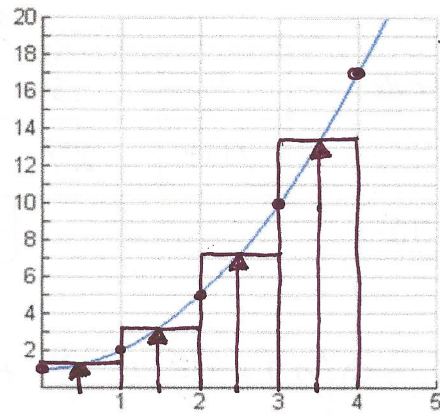


Use **TRAP₄** to approximate $\int_0^4 (x^2+1)dx$

$TRAP_4 = \left(\frac{1}{2}\right)(1)\left(1+2(2+5+10)+17\right)$
$= 26$

Under or Over estimate?

Figure 4: $f(x)=x^2+1$



x	y
$\frac{1}{2}$	$\frac{5}{4} \sim 1.25$
$\frac{3}{2}$	$\frac{13}{4} \sim 3.25$
$\frac{5}{2}$	$\frac{29}{4} \sim 7.25$
$\frac{7}{2}$	$\frac{53}{4} \sim 13.25$

Use **MID₄** to approximate $\int_0^4 (x^2+1)dx$

$MID_4 = (1)(1.25 + 3.25 + 7.25 + 13.25)$
$= 25$

Under or Over estimate?

Find the average of **LHS₄** & **RHS₄**. $\frac{18+34}{2} = \frac{52}{2} = 26 = TRAP_4$

Can you explain algebraically and geometrically why the average is equal to **TRAP₄** approximation.

EQUAL
$TRAP_4 = \left(\frac{1}{2}\right)(1)\left((1+2)+(2+5)+(5+10)+(10+17)\right)$
$\frac{LHS+RHS}{2} = \frac{(1+2+5+10)+(2+5+10+17)}{2}$

MIDLINE OF EACH TRAP are the average of the two bases.

KEY

MATH ▶ FRAC

§5.2 Definite Integral & Riemann Sums

Find an approximation for the definite integral by using Riemann sums with 4 subintervals using left endpoints, right endpoints, midpoints and the trapezoidal rule.

1. $\int_1^2 \frac{1}{x^2}\,dx$

$Y_1 = \frac{1}{x^2}$

	1.125	1.375	1.625	1.875	
x	1	1.25	1.50	1.75	2
$f(x)$	1	$\frac{16}{25}$	$\frac{4}{9}$	$\frac{16}{49}$	$\frac{1}{4}$
	1	$\boxed{\frac{64}{81}}$	$\boxed{\frac{64}{121}}$	$\boxed{\frac{64}{169}}$	$\boxed{\frac{64}{225}}$

$\Delta x =$ 0.25

$(Y_1(1) + Y_1(1.25) + Y_1(1.50) + Y_1(1.75))$

LHS$_4$ = $(0.25)\left(1 + \frac{16}{25} + \frac{4}{9} + \frac{16}{49}\right) = 0.6027437642$ 0.602 0.603

RHS$_4$ = $(0.25)\left(\frac{16}{25} + \frac{4}{9} + \frac{16}{49} + \frac{1}{4}\right) = 0.4152437642$ 0.415

☐ MID$_4$ = $(0.25)\left(\frac{64}{81} + \frac{64}{121} + \frac{64}{169} + \frac{64}{225}\right) = 0.4955479365$ 0.495 0.496

TRAP$_4$ = $\left(\frac{1}{2}\right)(.25)\left(1 + 2\left(\frac{16}{25} + \frac{4}{9} + \frac{16}{49}\right) + \frac{1}{4}\right) = 0.5089937642$ 0.508 0.509

2. $\int_3^6 3x^2\,dx$

	3.375	4.125	4.875	5.625	
x	3	3.75	4.50	5.25	6
$f(x)$	27	$\frac{675}{16}$	$\frac{243}{4}$	$\frac{1323}{16}$	108
		$\boxed{\frac{2187}{64}}$	$\boxed{\frac{3267}{64}}$	$\boxed{\frac{4563}{64}}$	$\boxed{\frac{6075}{64}}$

$\Delta x =$ 0.75

LHS$_4$ = $(0.75)\left(27 + \frac{675}{16} + \frac{243}{4} + \frac{1323}{16}\right) = 159.46875$ 159.468 159.469

RHS$_4$ = $(0.75)\left(\frac{675}{16} + \frac{243}{4} + \frac{1323}{16} + 108\right) = 220.21875$ 220.218 220.219

☐ MID$_4$ = $(0.75)\left(\frac{2187}{64} + \frac{3267}{64} + \frac{4563}{64} + \frac{6075}{64}\right) = 188.578125$ 188.578

TRAP$_4$ = $\frac{1}{2}(0.75)\left(27 + (2)\left(\frac{675}{16} + \frac{243}{4} + \frac{1323}{16}\right) + 108\right) = 189.84375$ 189.843 189.844

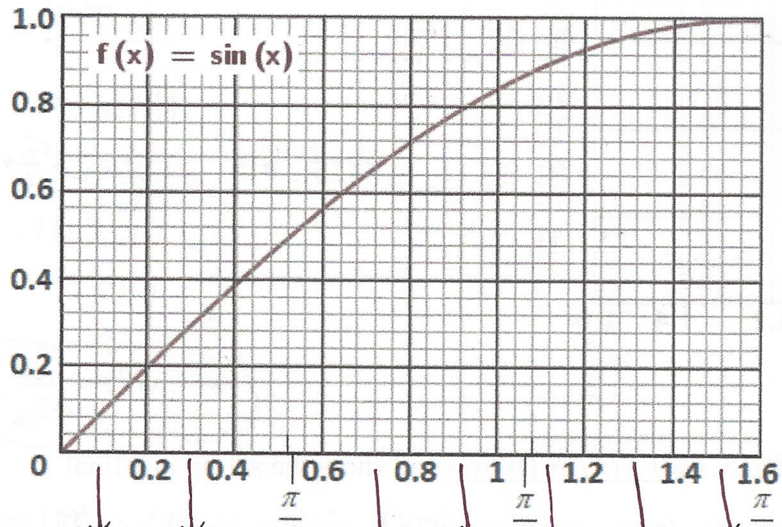
KEY

DAY 85

§5.2 An Introduction to the Definite Integral – Student Notes

Directions: Use the graphs for $f(x)=\sin(x)$ shown on the next page for the questions which follow.

1. Estimate the area under the graph of the sine curve over the interval $\left[0,\frac{\pi}{2}\right]$ by counting blocks. Show your work.

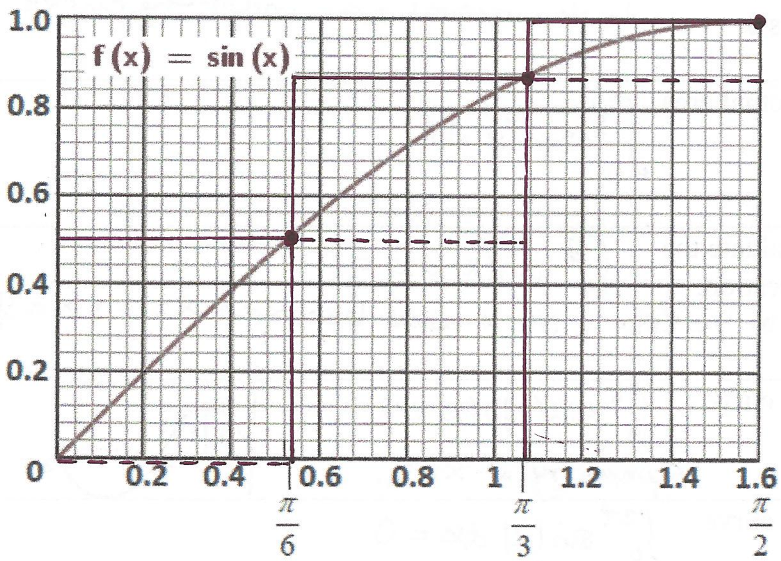


0.02	0.04	0.08	.12	.12	.16	.16	.16
	0.02	0.004	.0096	.024	.008	.024	.036
			.0008	.012	.008	.007	.002
	0.06	0.084	0.1304	0.156	0.156	0.191	0.198

0.06
0.084
0.1304
0.156
0.156
0.191
0.198

1.0154 square units of area between the curve $y=\sin(x)$ and the x-axis from $x=0$ to $x=\frac{\pi}{2}$.

2. Now, use 3 partitions find a left hand sum **LHS$_3$** and a right hand sum **RHS$_3$.** Shade your LHS in blue and RHS in red. Show computations in the space provided below. Average LHS$_3$ and RHS$_3$ to improve your estimate of the area under the graph of the sine curve. This average is the **TRAP$_3$**



x	$\sin(x)$
0	0
$\frac{\pi}{6}$	$\frac{1}{2}$
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$
$\frac{\pi}{2}$	1

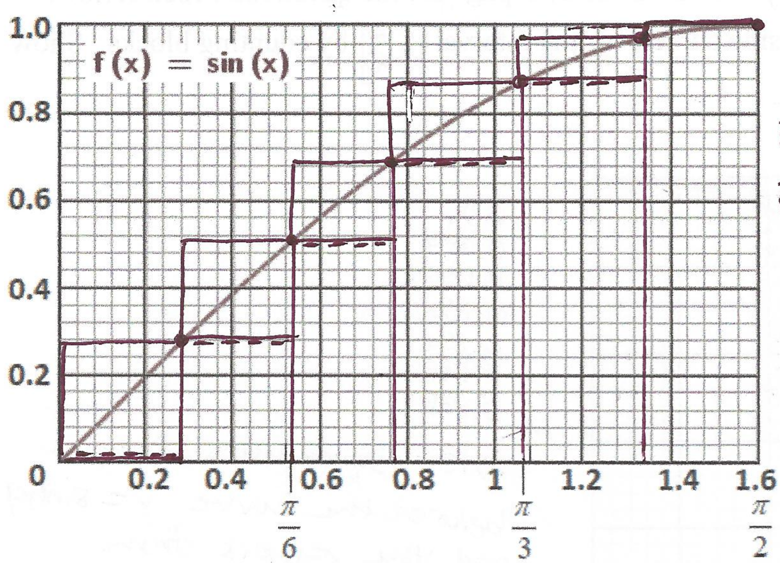
$LHS_3 = \frac{\pi}{6}\left(0+\frac{1}{2}+\frac{\sqrt{3}}{2}\right) = 0.715249\ldots = 0.715$

$RHS_3 = \frac{\pi}{6}\left(\frac{1}{2}+\frac{\sqrt{3}}{2}+1\right) = 1.238848\ldots = 1.238$ or 1.239

$TRAP_3 = \frac{LHS+RHS}{2} = 0.977048\ldots = 0.977$

KEY

3. Now, use 6 partitions find **LHS_6** and **RHS_6**. Shade the LHS in blue and the RHS in red. Show computations in the space below.



	x	$f(x)=\sin(x)$
	0	0
	$\frac{\pi}{12}$	$0.258819\ldots \xrightarrow{STO} A$
$\frac{2\pi}{12}$	$\frac{\pi}{6}$	$\frac{1}{2}$
$\frac{3\pi}{12}$	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$
$\frac{4\pi}{12}$	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$
$\frac{5\pi}{12}$	$\frac{5\pi}{12}$	$0.965925\ldots \xrightarrow{STO} B$
$\frac{6\pi}{12}$	$\frac{\pi}{2}$	1

$LHS_6 = \frac{\pi}{12}\left(0+A+\frac{1}{2}+\frac{\sqrt{2}}{2}+\frac{\sqrt{3}}{2}+B\right)$
$= 0.863382\ldots$
$= 0.863$

$RHS_6 = \frac{\pi}{12}\left(A+\frac{1}{2}+\frac{\sqrt{2}}{2}+\frac{\sqrt{3}}{2}+B+1\right)$
$= 1.125181\ldots$
$= 1.125$

Why is this *not* less than RHS_3?

4. Finally, use 12 partitions find **LHS_{12}** and **RHS_{12}** *but NO DRAWING* and show the set-up and final answer only. Use your calculator!

$LHS_{12} = \frac{\pi}{24}\left(0+\sin\left(\frac{\pi}{24}\right)+\sin\left(\frac{\pi}{12}\right)+\sin\left(\frac{\pi}{8}\right)+\sin\left(\frac{\pi}{6}\right)+\sin\left(\frac{5\pi}{24}\right)+\sin\left(\frac{\pi}{4}\right)+\sin\left(\frac{7\pi}{24}\right)+\sin\left(\frac{\pi}{3}\right)+\sin\left(\frac{3\pi}{8}\right)+\sin\left(\frac{5\pi}{12}\right)+\sin\left(\frac{11\pi}{24}\right)\right)$
≈ 0.93312185 $\quad 0.933$

$RHS_{12} = \frac{\pi}{24}\left(\sin\left(\frac{\pi}{24}\right)+\sin\left(\frac{\pi}{12}\right)+\sin\left(\frac{\pi}{8}\right)+\sin\left(\frac{\pi}{6}\right)+\sin\left(\frac{5\pi}{24}\right)+\sin\left(\frac{\pi}{4}\right)+\sin\left(\frac{7\pi}{24}\right)+\sin\left(\frac{\pi}{3}\right)+\sin\left(\frac{3\pi}{8}\right)+\sin\left(\frac{5\pi}{24}\right)+\sin\left(\frac{11\pi}{24}\right)+1\right)$
≈ 1.064021545 $\quad 1.064$

5. What seems to be happening to the estimate for the area as you increase the number of partitions used in calculating the Riemann Sums in the previous 3 questions?

As Δx decreases & number of rectangles increases the estimates get closer & closer to the actual area under the curve

6. Write a definite integral to represent the actual area under the sine curve over the interval $\left[0, \frac{\pi}{2}\right]$.

$\int_0^{\frac{\pi}{2}} \sin(x)\,dx$

7. Use the calculator feature MATH 9: to evaluate the definite integral you wrote in the previous question.

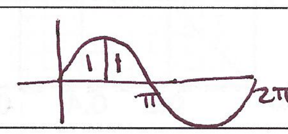
$\int_0^{\frac{\pi}{2}} \sin(x)\,dx = 1$

The area bounded by the function $y=\sin x$ & the x-axis between $x=0$ & $x=\frac{\pi}{2}$.

8. What do you think the area under the sine curve would be over the interval $[0,\pi]$?
Verify using MATH 9:

$\int_0^{\pi} \sin(x)\,dx = 2$

Symmetry at $x=\frac{\pi}{2}$.



9. What do you think the area under the sine curve would be over the interval $[0,2\pi]$.

$\int_0^{2\pi} \sin(x)\,dx = 0$

Someone may say 4...

10. Find the area under the sine curve over the interval $[0,2\pi]$ using MATH 9:
What happens?????? Why?

$\int_0^{2\pi} \sin(x)\,dx = 0$

area above the x-axis & below the curve is positive but area below the x-axis and above the curve is negative $\therefore$ areas cancel out.

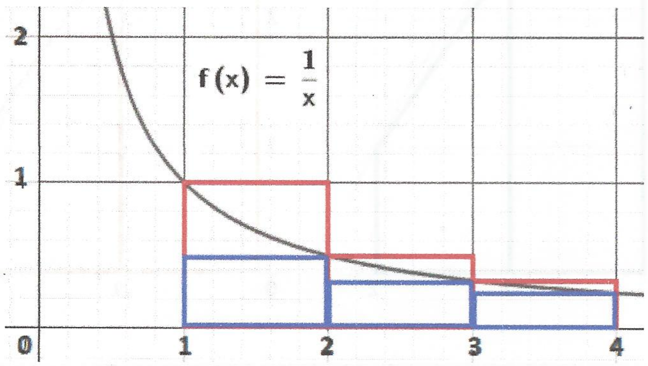
Key

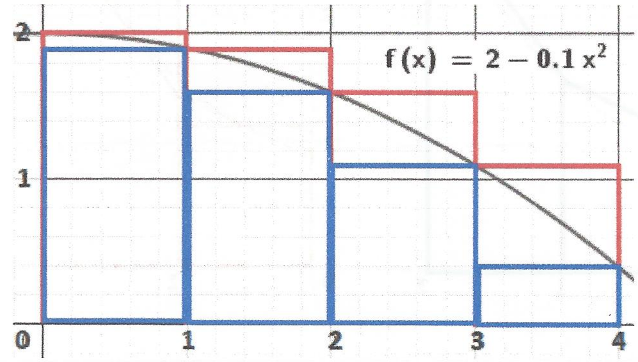
§5.2 & §7.5 Summary of Under/Over- Estimates of Numerical Approximations – Student Notes

For the ***LHS & RHS*** the approximated integral's relation to the exact integral value depends on whether the curve is ***increasing*** or ***decreasing***.

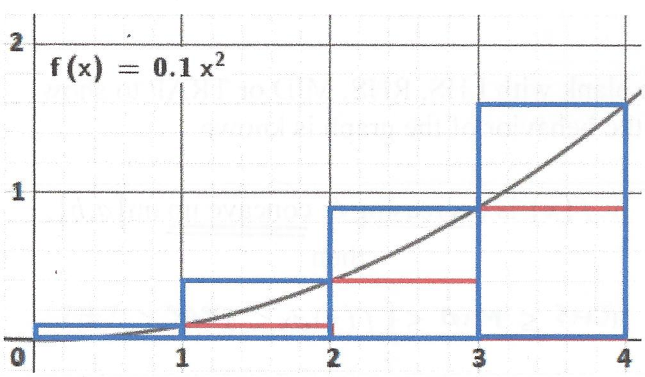
Examine the four functions below which illustrate the four combinations of increasing/decreasing and concave up/down functions. Red rectangles are used for LHS. Blue rectangles are used for RHS.

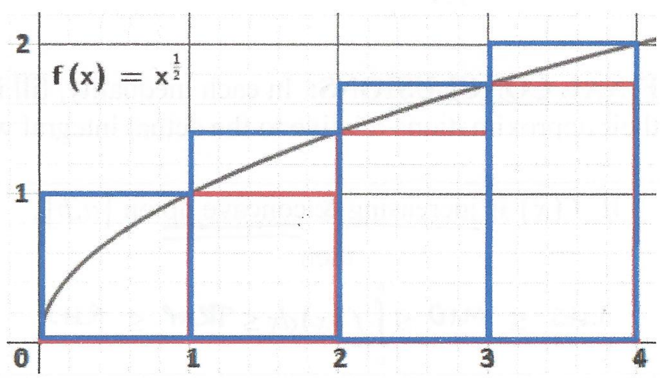
Complete each statement to show the LHS or RHS relative to the actual integral value.





If $f(x)$ is decreasing on $[1,4]$, then $\underline{\text{RHS}} \le \int_1^4 \frac{1}{x}\,dx \le \underline{\text{LHS}}$	If $f(x)$ is decreasing on $[0,4]$, then $\underline{\text{RHS}} \le \int_0^4 2-0.1x^2\,dx \le \underline{\text{LHS}}$





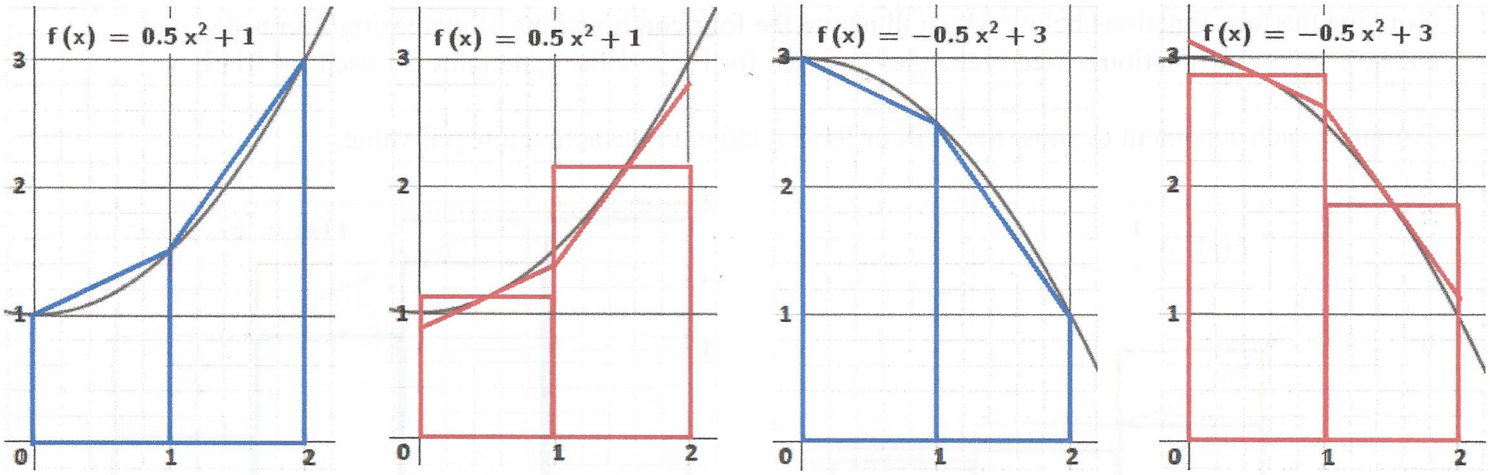
If $f(x)$ is increasing on $[0,4]$, then $\underline{\text{LHS}} \le \int_0^4 0.1x^2\,dx \le \underline{\text{RHS}}$	If $f(x)$ is increasing on $[0,4]$, then $\underline{\text{LHS}} \le \int_0^4 \sqrt{x}\,dx \le \underline{\text{RHS}}$

CONCLUSIONS:

If $f(x)$ is increasing on $[a,b]$, then $\underline{\text{LHS}} \le \int_a^b f(x)\,dx \le \underline{\text{RHS}}$	If $f(x)$ is decreasing on $[a,b]$, then $\underline{\text{RHS}} \le \int_a^b f(x)\,dx \le \underline{\text{LHS}}$

- For the ***TRAP & MID*** the approximated integral's relation to the exact integral value depends on whether the curve is ***concave up*** or ***concave down***.

Blue trapezoids are used for TRAP.
Red rectangles with tangent segments at midpoints are used for MID.



If $f(x)$ is concave up on $[a,b]$, then MID $\le \int_a^b f(x)\,dx \le$ TRAP	If $f(x)$ is concave down on $[a,b]$, then TRAP $\le \int_a^b f(x)\,dx \le$ MID

* MID & TRAP ARE CLOSER ESTIMATES TO INTEGRAL THAN LHS & RHS.

FINAL CONCLUSIONS: In each inequality, fill in the blank with LHS, RHS, MID or TRAP to show their approximations relative to the actual integral when the behavior of the graph is known.

If $f(x)$ is increasing & concave up on $[a,b]$, then LHS $\le$ MID $\le \int_a^b f(x)\,dx \le$ TRAP $\le$ RHS	If $f(x)$ is decreasing & concave up on $[a,b]$, then RHS $\le$ MID $\le \int_a^b f(x)\,dx \le$ TRAP $\le$ LHS
If $f(x)$ is increasing & concave down on $[a,b]$, then LHS $\le$ TRAP $\le \int_a^b f(x)\,dx \le$ MID $\le$ RHS	If $f(x)$ is decreasing & concave down on $[a,b]$, then RHS $\le$ TRAP $\le \int_a^b f(x)\,dx \le$ MID $\le$ LHS