§3.7 Implicit Differentiation -- A Brief Introduction -- Student Notes

Find these derivatives of these functions:

$y = \tan(x)$	$y = \sin(x)$	$y = e^x$
$\frac{d}{dx}(\tan(x)) =$	$\frac{d}{dx}(\sin(x)) =$	$\frac{d}{dx}(e^x) =$

Write the inverses of these functions:

$y = \tan(x)$	$y = \sin(x)$	$y = e^x$

How would we find the derivatives of these inverse functions?

$\frac{d}{dx}(\_\_\_\_\_\_\_\_\_\_\_\_)$	$\frac{d}{dx}(\_\_\_\_\_\_\_\_\_\_\_\_)$	$\frac{d}{dx}(\_\_\_\_\_\_\_\_\_\_\_\_)$

Let's look at a brief introduction to Implicit Differentiation so that we can find the derivatives of these three inverse functions.

Up to now, we have worked explicitly, solving an equation for one variable *y* in terms of another variable *x*. For example, if you were asked to find $\frac{dy}{dx}$ for $2x^2 + y^2 = 4$, you would solve for *y* and get $y = \pm\sqrt{4 - 2x^2}$ and then take the derivative. This derivative requires the use of the chain rule.

Sometimes it is inconvenient, difficult or impossible to solve for *y*. In this case, we use implicit differentiation. It is imperative to note that anytime you see a *y*-variable you must think of *y* as a function of *x* just as in the notation: $y = f(x)$. Since I do not know the explicit form of *f(x)* I will apply the chain rule to indicate it's derivative.

Differentiating with respect to *x*:

$$\frac{d}{dx}\left[x^3\right] = 3x^2$$

variables agree

Variables agree $\Rightarrow$ use power rule

$$\frac{d}{dx}\left[y^3\right] = 3y^2\frac{dy}{dx}$$

variables disagree

Variables disagree $\Rightarrow$ use power rule and chain rule

$$\frac{d}{dx}\left[2x^5 + 3y\right] = 10x^4 + 3\frac{dy}{dx}$$

variables disagree

variables agree

Practice: 1. $-3y^2 = x^4 + 5$ 2. $\sqrt{y} + y^2 = \cos\left(x^3\right)$

Find $\dfrac{dy}{dx}$

§3.6 Implicit Differentiation -- Student Notes

Let's return to our 3 inverse functions and use implicit differentiation to find their derivatives:

$\dfrac{d}{dx}\left(\arctan(x)\right)$ $\dfrac{d}{dx}\left(\arcsin(x)\right)$ $\dfrac{d}{dx}\left(\ln(x)\right)$

Derivatives of some important Inverse functions (MEMORIZE THESE).

$\dfrac{d(\arctan x)}{dx} =$	$\dfrac{d(\arcsin x)}{dx} =$	$\dfrac{d(\ln x)}{dx} =$

Note: $\arctan x = \tan^{-1} x$ & $\arctan(\tan x) = \tan(\arctan x) = x$
$\arcsin x = \sin^{-1} x$ & $\arcsin(\sin x) = \sin(\arcsin x) = x$

Examples using the derivative rules we just found and applying rules we already learned:

a) $\dfrac{d(\arctan(t^2))}{dt}$ b) $\dfrac{d(\arcsin(\tan(\theta)))}{d\theta}$ c) $\dfrac{d\ln(x^2+1)}{dx}$

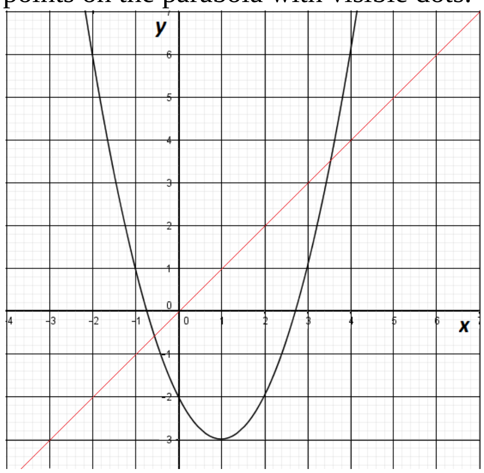
d) $\dfrac{d(t^2 \ln t)}{dt}$ e) $\dfrac{d(\sqrt{1+\ln(2y)})}{dy}$ f) $\dfrac{d(\cos(\sin^{-1} x))}{dx}$

Derivative of Inverse Function Theorem

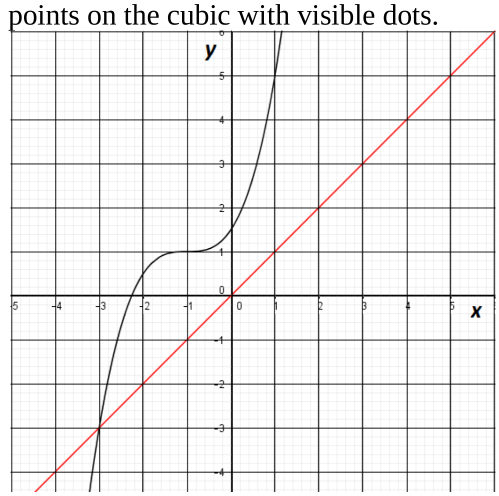
Function and Inverse Pre-requisites:

Given each function, identify key points on the function that fall on lattice points of the coordinate grid.

Mark 7 points on the parabola with visible dots.

1) 

Mark 3 points on the cubic with visible dots.

2) 

a) Write the equation of each function in (h,k) form and evaluate the function at the given point.

Quadratic function $f(x) =$	$(3, f(3)) =$	Cubic function $f(x) =$	$(1, f(1)) =$

b) For each function, list the operations on *x* that yield y.

Parabola

Cubic

c) Write inverse equations by using the list in (b) & applying *inverse operations in reverse order on x*.
State the corresponding inverse coordinate from the point on the function in part (a)

$f^{-1}(x) =$	$(x, f^{-1}(x)) =$	$f^{-1}(x) =$	$(x, f^{-1}(x)) =$

d) Accurately, sketch the inverse function on the coordinate grid using the key lattice points. Label $f^{-1}(x)$.

e) Find the derivative of the function at the specified point.
Find the derivative of its inverse at the corresponding point on the inverse.

$f'(x) =$	$\left.\frac{dy}{dx}\right\|_{x=3} =$	$f'(x) =$	$\left.\frac{dy}{dx}\right\|_{x=1} =$
$(f^{-1})'(x) =$	$(f^{-1})'(\_\_\_\_) =$	$(f^{-1})'(x) =$	$(f^{-1})'(\_\_\_\_) =$

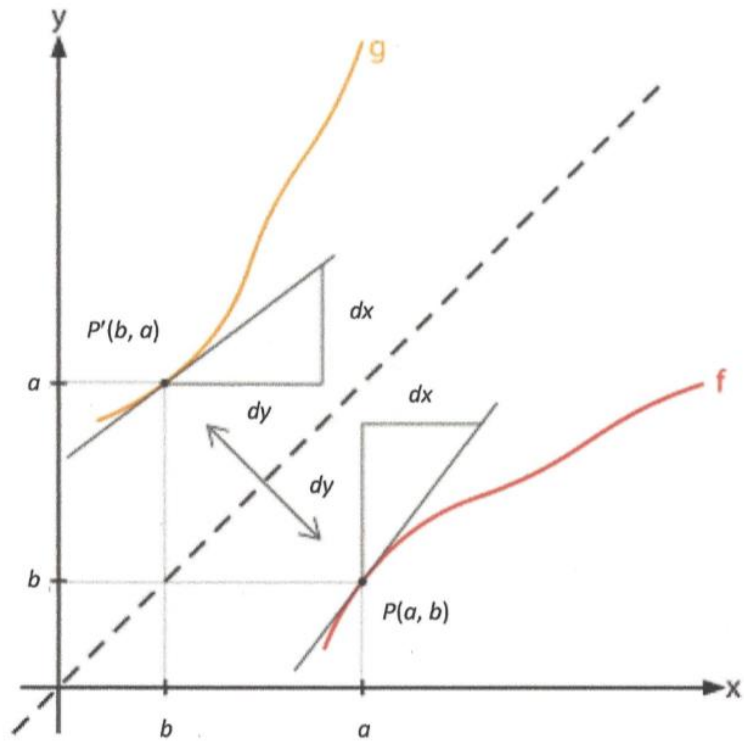
f) What is the relationship between the derivative value of the function at the point and its inverse at the corresponding inverse point? ____

AB Calculus Supplement: Derivative of the Inverse of a Function

Suppose that f and $g = f^{-1}$ are inverse functions. What is the relationship between their derivatives?

- Algebraically: inverses are obtained by interchanging the x and y coordinates and solving for y.
- Graphically: inverses are reflections of each other in the line $y = x$.

If f passes through the point (a,b), then the slope of the curve at $x = a$ is represented by $f'(a)$ and by the ratio of the change in y over the change in x, $\frac{\Delta y}{\Delta x}$. In the figure, note the slope triangle at point $P(a,b)$ on f with vertical length dy and horizontal length dx and slope $\frac{dy}{dx}$.



When the graph of f is reflected in the line $y = x$, we obtain the graph of the inverse of f denoted as f^{-1} and this inverse graph passes through the point (b,a). The slope of the inverse curve at $x = b$ is represented by $(f^{-1})'(b)$ and by the ratio of the change in x over the change in y, $\frac{\Delta x}{\Delta y} = \frac{dx}{dy}$ because the horizontal and vertical lengths of the slope triangle at Point P were interchanged at P'.

The slope of the line tangent to the graph of $g = f^{-1}$ at $x = b$ is the reciprocal of the slope of f at $x = a$.

http://demo.activemath.org/ActiveMath2/LeAM_calculatorPics/DerivInverseFunction.png?lang=en

Derivative of the Inverse of a Function

Given (a,b) is a point on f and $g = f^{-1}$ is the inverse of f,

if $f'(a) = m$, then $g'(b) = (f^{-1})'(b) = \frac{1}{m}$.

The derivative of the inverse of a function at a point is
the reciprocal of the derivative of the function at the corresponding point.

Examples:

1) If $f(7)=1$ and $f'(7)=5$ and g is the inverse of f, that is $g=f^{-1}$, then $g'(1)=?$

2) If $f(-2)=5$, $f'(-2)=6$ and $f'(5)=-3$ and g is the inverse of f, that is $g=f^{-1}$, then $g'(5)=?$

3) Values for a function f and its derivative are shown in the table. If g is the inverse of f then evaluate $g'(4)$ and $g'(-1)$.

x	$f(x)$	$f'(x)$
-3	4	0.25
2	-1	$\frac{-2}{3}$

4) Let $f(x)=\sqrt{x}$, and let g be the inverse of f. Evaluate $g'(3)$.

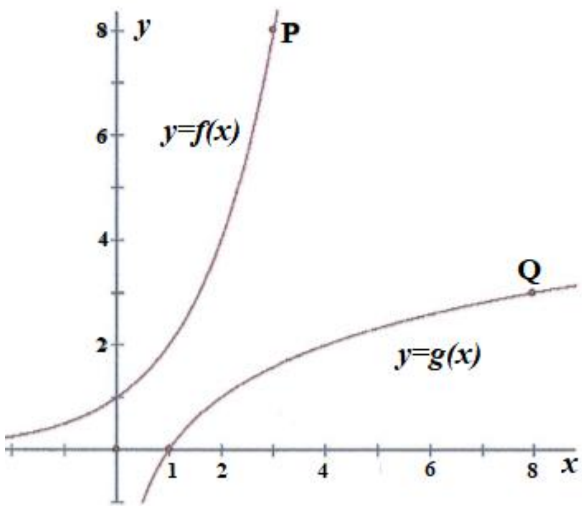
5) Let $f(2)=-3$, $f'(2)=\frac{3}{4}$ & g be the inverse of f, what is the equation of the tangent line to $g(x)$ at $x=-3$?

6) The following figure shows $f(x)$ and $f^{-1}(x)$. Using the given table, find:

x	$f(x)$	$f'(x)$
0	1	0.7
1	2	1.4
2	4	2.8
3	8	5.5

a) $f(2)$, $f^{-1}(2)$, $f'(2)$, $(f^{-1})'(2)$.

b) The equation of the tangent line at the points $P(3,8)$ and $Q(8,3)$.



c) What is the relationship between the two tangent lines?

7) Calculate $g'(1)$ where $g(x)$ is the inverse of $f(x)=x+e^x$ without solving for $g(x)$.

8) Calculate $g'(x)$ where $g(x)$ is the inverse of $f(x)=x^3+1$ without solving for $g(x)$.

9) Let $f(x)=\frac{1}{4}x^3+x-1$. Assume $f(x)$ is one-to-one, meaning that $f(x)$ has an inverse that is also a function.

a) What is the value of $f^{-1}(x)$ when $x=3$?

b) Find the slope of the tangent line to the curve $y=f^{-1}(x)$ at $x=3$.

Keys to Properly Solving Derivative of Inverse Problems:

- Identify the point (a,b) on the function f using the information that is given.
- Differentiate f
- Take the reciprocal of the derivative of f. This is the derivative of f^{-1}.
- Evaluate the derivative of f^{-1} at the point (b,a)

Practice:
Given the table of values for differentiable functions f and g.

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
1	**2**	**½**	**-3**	**5**
2	**3**	**1**	**0**	**4**
3	**4**	**2**	**2**	**3**
4	**6**	**5**	**3**	**-½**

a) If $h(x) = f^{-1}(x)$, then evaluate $h'(4)$.

b) If $h(x) = f^{-1}(x)$, then evaluate $h'(2)$.

c) If $d(x) = g^{-1}(x)$, then evaluate $d'(-3)$.

d) If $d(x) = g^{-1}(x)$, then evaluate $d'(3)$.

And these are not derivatives of inverses, but they are good practice.

e) If $p(x) = g^2(x)$, then evaluate $p'(3)$.

f) If $b(x) = f(x)\bullet g(x)$, then evaluate $b'(2)$.

g) If $n(x) = f(x^3)$, then evaluate $p'(1)$.

h) If $m(x) = f\left(\frac{x}{3}\right)$, then evaluate $m'(9)$.

i) If $q(x) = g(2x+3)$, then evaluate $q'\left(\frac{1}{2}\right)$.

§3.7 Implicit Differentiation -- Student Notes

Up to now, we have worked explicitly, solving an equation for one variable in terms of another. For example, if you were asked to find $\frac{dy}{dx}$ for $2x^2 + y^2 = 4$, you would solve for *y* and get $y = \pm\sqrt{4-2x^2}$ and then take the derivative. Sometimes it is inconvenient or difficult to solve for *y*. In this case, we use implicit differentiation. You assume *y* could be solved in terms of *x* and treat it as a function in terms of *x*. Thus, you must apply the chain rule because you are assuming *y* is defined in terms of *x*.

Differentiating with respect to *x*:

$$\frac{d}{dx}\left[x^3\right] = 3x^2$$

variables agree

Variables agree $\Rightarrow$ use power rule

$$\frac{d}{dx}\left[y^3\right] = 3y^2\frac{dy}{dx}$$

variables disagree

Variables disagree $\Rightarrow$ use power rule and chain rule

$$\frac{d}{dx}\left[x+3y\right] = 1 + 3\frac{dy}{dx}$$

variables disagree

variables agree

$$\frac{d}{dx}\left[xy^2\right] = 1\left(y^2\right) + x\left(2y\frac{dy}{dx}\right) = y^2 + 2xy\frac{dy}{dx}$$

variables disagree

use product and chain rules

Consider the problem, find $\frac{dy}{dx}$ for $y^2 - 2y + 3x = x^2$. Treat *y* as a quantity in terms of *x* so

$$\frac{d}{dx}\left(y^2\right) - \frac{d}{dx}\left(2y\right) + \frac{d}{dx}\left(3x\right) = \frac{d}{dx}\left(x^2\right)$$

Different Same

$$2y\frac{dy}{dx} \qquad -2\frac{dy}{dx} \qquad +3 \qquad = 2x$$

Now solve for $\frac{dy}{dx}$. $2y\frac{dy}{dx} - 2\frac{dy}{dx} = 2x - 3 \Rightarrow \frac{dy}{dx}\left(2y-2\right) = 2x - 3 \Rightarrow \frac{dy}{dx} = \frac{2x-3}{2y-2}$

Guidelines for Implicit Differentiation:

1. Differentiate both sides of the equation with respect to *x*.
2. Collect all terms involving $\frac{dy}{dx}$ on one side of the equation and move all other terms to the other side.
3. Factor $\frac{dy}{dx}$ out of the terms on the one side.
4. Solve for $\frac{dy}{dx}$ by dividing both sides of the equation by the factored term.

Practice:

Find $\frac{dy}{dx}$:

1. $y^3 + 7\cos(y) = x^3$

2. $4x^2 y - 3y = x^3 - 1$

3. $x^2 + 5y^3 = xy + 9$

4. $x^3 + x^2 y - 10y^4 = 0$

5. Find the equation of the normal line (the line perpendicular to the tangent line) to the curve $2\left(x^2 + y^2\right)^2 = 25\left(x^2 - y^2\right)$ at the point (3, 1).

6 Given $y = f^{-1}(x)$ rewrite the inverse function as $f(y) = x$ then use implicit differentiation to find the derivative of an inverse function. Write the final answer in terms of only x.

§3.9 Linear Approximation and the Derivative – Student Notes

Tangent Line Approximations:
We can use the equation of the tangent line to approximate the value of a function at a particular value of x.

The concavity of the function tells us if an approximation made with the tangent line if an over-estimate (too high) or an under-estimate (too low.)

If a function is concave up, the tangent line will be ___________ the curve and any approximation made from the tangent line equation will be ________________.

If a function is concave down, the tangent line will be _________ the curve and any approximation made from the tangent line equation will be ________________.

Sketch four portions of graphs satisfying the criteria given, then draw a point on each of the portions and draw a tangent line to the curve at that point. Do your pictures illustrate the conclusions you made above?

$f'>0$ & $f''>0$	$f'>0$ & $f''<0$	$f'<0$ & $f''>0$	$f'<0$ & $f''<0$

For each question below, write the equation of the tangent line to the curve at the designated value of x. Use the tangent line equation to approximate the value of the function at the given x-value. Finally use the 2nd Derivative and concavity to justify whether the tangent line approximation is too high or too low.

Function & $x=a$	Tangent line equation at $x=a$	Tangent line approximation at $x=a$	Second Derivative evaluated at $x=a$	Is the tangent line approximation an overestimate or underestimate? Justify using f”
1 $f(x)=\sqrt{x}$ $x=49$		$f(50)\approx$	$f''(49)=$	
2 $f(x)=\dfrac{1}{x}$ $x=1$		$f(1.1)\approx$	$f''(1)=$	

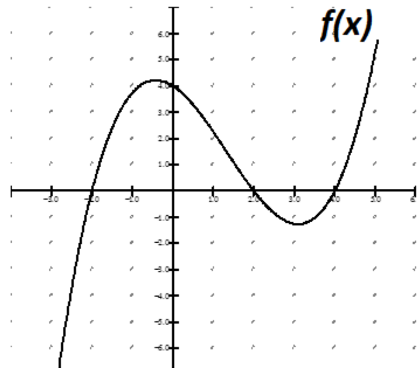
3 $f(x)=\ln(x)$ $x=e$		$f(3)\approx$	$f''(e)=$	
4 $g(x)=\frac{1}{\sqrt{1+x}}$ $x=1$		$f(1.1)\approx$	$f''(1)=$	
5 $h(x)=\frac{1}{1+x^2}$ $x=1$		$f(1.01)\approx$	$f''(1)=$	
6 $j(x)=\cos(x)$ $x=\frac{\pi}{6}$		$f(0.5)\approx$	$f''\left(\frac{\pi}{6}\right)=$	

§3.10 Mean Value Theorem – Student Notes

MEAN VALUE THEOREM: If a function is <u>continuous</u> on [a, b] and <u>differentiable</u> on (a, b), then there is as number c in the interval (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a} \quad \textbf{or} \quad (b-a)\, f'(c) = f(b) - f(a).$$

1. Use the graph to illustrate the Mean Value Theorem with a continuous and differentiable function. Show *f (x), a, b, c* and all other conditions of the theorem.



2. Find the number c that satisfies the Mean Value Theorem (MVT) for $f(x) = \sqrt{x}$ on the interval [0,4]. Draw a picture.

3. Why does the MVT not apply?

a) $y = \dfrac{x+3}{x-2}$ on [0,3]

b) $f(x) = x^{\frac{1}{3}}$ on [-1,1]

4. Apply the MVT, if possible. If not possible explain why.

A $f(x) = x^2$ on [-2,1]	B $f(x) = x^3 - 3x^2$ on [0,3]	C $f(x) = x^{\frac{2}{3}}$ on [0,1]

MVT Problems

1. The function $f(x)=x^{\frac{2}{3}}$ on [- 8, 8] does not satisfy the conditions of the Mean Value Theorem because
   A. *f (0)* is not defined
   B. *f (x)* is not continuous of [-8, 8]
   C. $f'(-1)$ does not exist
   D. *f (x)* is not defined for x < 0.
   E. $f'(0)$ does not exist

2. If *f (a)* = *f (b)* and *f (x)* is continuous on [a, b], then
   A. *f (x)* must be identically zero
   B. $f'(x)$ may be different from zero for all *x* on [a,b]
   C. there exists at least one number c, a < c< b, such that $f'(c)=0$
   D. $f'(x)$ must exist for every *x* on (a, b)
   E. none of the preceding is true

3. Find the value of *c* that satisfies the Mean Value Theorem for $f(x)=x^3+x-4$ on the interval [- 2, 1].
   A. -1 B. 1 C. 0 D. 4 E. None of these.

4. Find the number that satisfies the MVT on the given interval **<u>or</u>** state why the theorem does not apply.
   a) $f(x)=x^{\frac{2}{5}}$ on [0, 32]
   b) $f(x)=\dfrac{1}{(x-2)^2}$ on [2, 5]
   c) $g(x)=x+\dfrac{1}{x}$ on [1, 3]
   d) $h(x)=x^{\frac{1}{2}}+2(x-2)^{\frac{1}{3}}$ on [1, 9]

2003 #92: Let f be defined by $f\left(x\right)=x+\ln\left(x\right)$. What is the value of c for which the instantaneous rate of change of f at $x=c$ is the same as the average rate of change of f over $\left[1,4\right]$?

(A) 0.456 (B) 1.244 (C) 2.164 (D) 2.342 (E) 2.452

HW MVT

Write the definition of continuity. 1) 2) 3)

Write mathematical notation for differentiability:

State the two prerequisite conditions that must be determined before the Mean Value Theorem can be applied.

1) 2)

What two calculations must be determined before making a conclusion using the Mean Value Theorem.

1) 2)

Read questions #1-4. If the function satisfies the hypotheses of the Mean Value Theorem, then solve for the value of *c* that satisfies the conclusion of the Mean Value Theorem. Otherwise, tell why it fails to meet the conditions of the Mean Value Theorem.

1. Given $f(x) = 5 - \frac{4}{x}$, find all values, *c*, in the interval [1,4].

2. Given $f(x) = x^4 - 2x^2$, find all values, *c*, in the interval [-2,2].

3. Given $f(x) = x(x^2 - x - 2)$, find all values, *c*, in the interval [-1,1].

4. Given $f(x) = x^{\frac{2}{3}} - 1$, find all values, *c*, in the interval [-8,8].