UNIT 4 TEST REVIEW. #1–18 ** CALCULATOR

① $f(x) = x^3 - 2x + 1$ $\quad f(0) = 1 \quad f(2) = 5 \quad f(-3) = -20$

$f'(x) = 3x^2 - 2$ $\quad g(1) = 0 \quad g(5) = 2 \quad g(20) = -3$

$f(x)$ & $g(x)$ are inverses. $\quad f'(0) = -2 \quad f'(2) = 10 \quad f'(-3) = 25$

$g'(1) = -\frac{1}{2} \quad g'(5) = \frac{1}{10} \quad g'(20) = \frac{1}{25}$

** Find $g'(116) = ?$ ⟶ $f(x) = 116 = x^3 - 2x + 1$

$Y_1 = x^3 - 2x + 1$ or $Y_1 = x^3 - 2x - 115$

$Y_2 = 116$ find zero.

find intersection $x = 5$

$x = 5$

$(5, f(5)) = (5, 116)$

$f'(5) = 73$

$(116, g(116)) = (116, 5)$

$g'(116) = \frac{1}{73}$

② a) $(f^{-1})'(2) = \frac{1}{f'(3)} = \frac{-1}{8}$

$f(x) = 2 \rightarrow x = 3$

b) $(f^{-1})'(3) = \frac{1}{f'(4)} = 2$

$f(x) = 3 \rightarrow x = 4$

c) $(f^{-1})'(0) = \frac{1}{f'(2)} = \frac{3}{4}$

$f(x) = 0 \rightarrow x = 2$

d) $(f^{-1})'(-3) = \frac{1}{f'(1)} = \frac{1}{5}$

$f(x) = -3 \rightarrow x = 1$

③ $\frac{d}{dx}\left(x^2 + xy + y^3 = 0\right)$

$2x + 1y + x\frac{dy}{dx} + 3y^2\frac{dy}{dx} = 0$

$\frac{dy}{dx}(x + 3y^2) = -(2x + y)$

$\frac{dy}{dx} = \frac{-(2x+y)}{x + 3y^2}$

A

④ $x^2 + xy = 10$ & $x = 2$ ⟶ $4 + 2y = 10$

$y = \frac{6}{2} = 3$

$2x + 1y + x\frac{dy}{dx} = 0$

$\frac{dy}{dx} = \left.\frac{-(2x+y)}{x}\right|_{\substack{x=2 \\ y=3}} = \frac{-(4+3)}{2} = \frac{-7}{2}$

A

UNIT 4 TEST REVIEW:

5. $x^2+y^2=25$

$$2x+2y\frac{dy}{dx}=0$$

$$\frac{dy}{dx}=\frac{-2x}{2y}=\frac{-x}{y}$$

$$\frac{d^2y}{dx^2}=\frac{y(-1)-(-x)\frac{dy}{dx}}{y^2}$$

multiply by $\frac{LCD}{LCD}$ to eliminate complex fraction.

$$\frac{d^2y}{dx^2}=\frac{\left(-y+x\left(\frac{-x}{y}\right)\right)}{(y^2)}\left(\frac{y}{y}\right)\quad \left.\frac{-y^2-x^2}{y^3}\right|_{(4,3)}$$

$$\frac{d^2y}{dx^2}=\frac{-9-16}{27}=\frac{-25}{27}$$

A

6. $y^3-xy^2=4$ when $y=2 \rightarrow 8-4y=4$

$-4y=-4$

$y=1 \quad (1,2)$

$$3y^2\frac{dy}{dx}-\left(1y^2+x\cdot 2y\cdot\frac{dy}{dx}\right)=0$$

$$(3y^2-2xy)\frac{dy}{dx}=y^2$$

$$\frac{dy}{dx}=\left.\frac{y^2}{3y^2-2xy}\right|_{(1,2)}=\frac{4}{12-4}=\frac{1}{2}$$

TANGENT LINE:

$$\boxed{y=\frac{1}{2}(x-1)+2}$$

7. $\ln(xy)=x+y$

$$\frac{1}{xy}\left(1y+x\frac{dy}{dx}\right)=1+\frac{dy}{dx}$$

$$\frac{1}{x}+\frac{1}{y}\frac{dy}{dx}=1+\frac{dy}{dx}$$

$$\frac{1}{x}-1=\left(1-\frac{1}{y}\right)\frac{dy}{dx}$$

$$\frac{1-x}{x}=\left(\frac{y-1}{y}\right)\frac{dy}{dx}$$

$$\frac{dy}{dx}=\frac{y(1-x)}{x(y-1)}$$

8. $f(x)=\frac{1}{\sqrt{1-x}}=(1-x)^{-\frac{1}{2}}$

$$f'(x)=-\frac{1}{2}(1-x)^{-3/2}(-1)$$

$$f'(x)=\frac{1}{2(1-x)^{3/2}}$$

$$f''(x)=-\frac{3}{4}(1-x)^{-5/2}(-1)$$

$$f''(x)=\frac{3}{4(1-x)^{5/2}}$$

$(-3, f(-3))=\frac{1}{2} \qquad f'(-3)=\frac{1}{16}$

Tangent Line: $t(x)=\frac{1}{16}(x+3)+\frac{1}{2}$

$$f(-3.1)\approx t(-3.1)=\frac{1}{16}\left(\frac{-1}{10}\right)+\frac{1}{2}=\frac{-1}{160}+\frac{80}{160}=\frac{79}{160}$$

$$f''(-3)=\frac{3}{4(1+3)^{5/2}}=\frac{3}{4(32)}=\frac{3}{128}>0$$

$\therefore$ $f(x)$ is concave up at $x=-3$

$t(-3.1)$ is an underestimate.

-3 -3.1

$f(-3.1)-t(-3.1)=0.0001147\ldots$

0.49386 − 0.49375

B/c f is above t(x), difference should be positive

UNIT 4 TEST REVIEW

(9) $f(x) = \ln(1-x)$

$f'(x) = \frac{-1}{1-x} = -1(1-x)^{-1}$

$f''(x) = +1(1-x)^{-2}(-1)$

$= \frac{-1}{(1-x)^2}$

$f''(x) < 0$ for all x

$\therefore$ $f(x)$ is always concave down

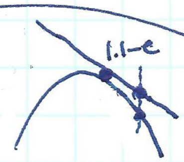
$(1-e, f(1-e)) = (1-e, 1)$ $\quad f'(1-e) = \frac{-1}{e}$

Tangent Line: $t(x) = \frac{-1}{e}(x-(1-e)) + 1$

$t(x) = -\frac{1}{e}(x+e-1) + 1$

$f(1.1-e) \approx t(1.1-e) = \frac{-1}{e}\left(\frac{1}{10}\right) + 1$

$= \frac{-1}{10e} + 1 \approx 0.96321$



$\therefore t(1.1-e)$ is an overestimate

$f(1.1-e) - t(1.1-e)$

$0.96251 - 0.96321 \approx -0.00069374$

B/c f is below $t(x)$ difference should be negative.

(10) $f(x) = \frac{1}{4}x^3 + 1 \quad [0,2]$

- $f(x)$ is continuous on $[0,2]$ b/c it is a polynomial & all polys are continuous.
- $f(x)$ is differentiable on $(0,2)$ b/c all polys are differentiable.

MVT: "slope of secant = slope of tangent"

- $\frac{f(2)-f(0)}{2-0} = f'(c) \qquad f'(x) = \frac{3}{4}x^2$

$\frac{(2+1)-(0+1)}{2-0} = \frac{2}{2} = 1 = \frac{3}{4}x^2$

$\frac{4}{3} = x^2$

$x = \frac{\pm 2}{\sqrt{3}} = \frac{\pm 2\sqrt{3}}{3} \approx \pm 1.154$

Since restricted domain is $[0,2]$ we will only consider $\frac{+2\sqrt{3}}{3}$

So $c = \frac{+2\sqrt{3}}{3}$ satisfies the MVT

(11) Since you traveled 5 miles in 5 minutes the average rate of change (slope of the secant) over the 5 minutes of driving time is $\frac{5 \text{ miles}}{\frac{1}{12} \text{ hr}} = 60$ mph.

The mean value theorem guarantees the police that your instantaneous velocity was 60 mph at least once while you drove that 5 minute section of highway.

UNIT 4 TEST REVIEW

(12) $f'(x)$ graph is given

a) critical points $f'(x) = 0$

$x = -2, 1, 3$

f': $+$ 0 $+$ 0 $-$ 0 $+$ at -2, 1, 3 (↗ -2 ↗ 1 ↘ 3 ↗)

b) f is increasing on $(-\infty, 1)$ $(3, \infty)$ b/c $f'(x) > 0$.

decreasing on $(1, 3)$ b/c $f'(x) < 0$

c) $(-2, f(-2))$ is a terrace point b/c there is no change in sign of f'.

$(1, f(1))$ is a relative maximum b/c f' changes sign $\oplus$ to $\ominus$.

$(3, f(3))$ is a relative minimum b/c f' changes sign $\ominus$ to $\oplus$.

d) f is concave up $(-2, -0.2)$ $(2.2, \infty)$ b/c f' is increasing & $f'' > 0$.

f is concave down $(-\infty, -2)$ $(-0.2, 2.2)$ b/c f' is decreasing & $f'' < 0$.

(13) $f(x)$ has $f'(x) = 3(x+2)(x+1)^2(x-3)^3$

$f'(x)$ is a positive 6^{th} degree poly.

$f'(x)$: $+$ 0 $-$ 0 $-$ 0 $+$ at -2, -1, 3 (↗ -2 ↘ -1 ↘ 3 ↗)

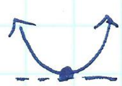
$f(x)$ has relative maximum $= f(-2)$ b/c $f'(x)$ changes $\oplus$ to $\ominus$.

$f(x)$ has relative minimum $= f(3)$ b/c $f'(x)$ changes sign $\ominus$ to $\oplus$.

$f(x)$ has terrace point $(-1, f(-1))$ b/c $f'(x)$ does not change sign

(14) $f''(x) = -3x^2 + 6x + 1$ $f(x)$ has a critical point at $x = 1$.

$f''(1) = -3 + 6 + 1 = 4 > 0$



$f(1)$ is a relative minimum by the 2nd Deriv test.

b/c $f''(1) > 0$ & f is concave up.

(15) $f(x) = 2x^3 - 3x^2 - 12x$ $[-2, 4]$

$f'(x) = 6x^2 - 6x - 12$

$= 6(x^2 - x - 2)$

$= 6(x-2)(x+1) = 0$

$x = 2, -1$

f': $[$ $+$ 0 $-$ 0 $+$ $]$ on -2, -1, 2, 4 (↗ ↘ ↗)

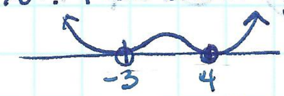
-2 MIN, -1 MAX, 2 MIN, 4 MAX.

$y = 2x^3 - 3x^2 - 12x$ factor to evaluate

x	$y = x(2x^2 - 3x - 12)$
-2	$-2(8+6-12) = -4$ = Rel min
-1	$-1(2+3-12) = 7$ = Rel MAX
2	$2(8-6-12) = -20$ = ABSOLUTE MIN
4	$4(32-24) = 32$ = ABSOLUTE MAX

UNIT 4 TEST REVIEW.

(16) $f(x) = \left(\frac{x-4}{x+3}\right)^2$ NOTE: $f(x) > 0$ for all x. Related Poly



a) $f'(x) = 2\left(\frac{x-4}{x+3}\right)^1 \left(\frac{x+3-(x-4)}{(x+3)^2}\right)$

$f'(x) = \frac{14(x-4)}{(x+3)^3}$

b) critical pts: $f' = 0$ $\quad$ f' und.

$x = 4$ $\quad$ $x = -3$

c) f': $+$ | $\emptyset$ at -3 (VA) | $-$ | 0 at 4 (MIN) | $+$

Related Poly $\oplus$ 4th deg.

d) $f''(x) = \frac{(x+3)^3(14) - 14(x-4)(3(x+3)^2)}{(x+3)^6}$

$f''(x) = \frac{(x+3)^2(14)\left[(x+3)-(x-4)(3)\right]}{(x+3)^6}$

$f''(x) = \frac{14(x+3-3x+12)}{(x+3)^4}$

$f''(x) = \frac{14(-2x+15)}{(x+3)^4}$

e) $f'' = 0$ $\quad$ f'' und $\quad$ $\ominus$ 5th deg Related poly

$x = \frac{15}{2} = 7.5$ $\quad$ $x = -3$

f) f'': $+$ | $\emptyset$ at -3 | $+$ | 0 at $\frac{15}{2}$ (INF. PT.) | $-$

g) f is increasing on $(-\infty, -3)\ (4, \infty)$
b/c $f' > 0$

f is decreasing on $(-3, 4)$
b/c $f' < 0$.

h) f is concave up on $(-\infty, -3)(-3, \frac{15}{2})$
b/c $f'' > 0$

f is concave down on $(\frac{15}{2}, \infty)$
b/c $f'' < 0$.

i) TABLE:

x	-3	4
y	und.	0

$f(4) = 0$ is an ABSOLUTE MINIMUM.
b/c f' changes sign $\ominus$ to $\oplus$
& lowest y-value on graph.

$f\left(\frac{15}{2}\right) = \left(\frac{\frac{15}{2}-4}{\frac{15}{2}+3}\right)^2 = \left(\frac{15-8}{15+6}\right)^2 = \left(\frac{7}{21}\right)^2 = \frac{1}{9}$

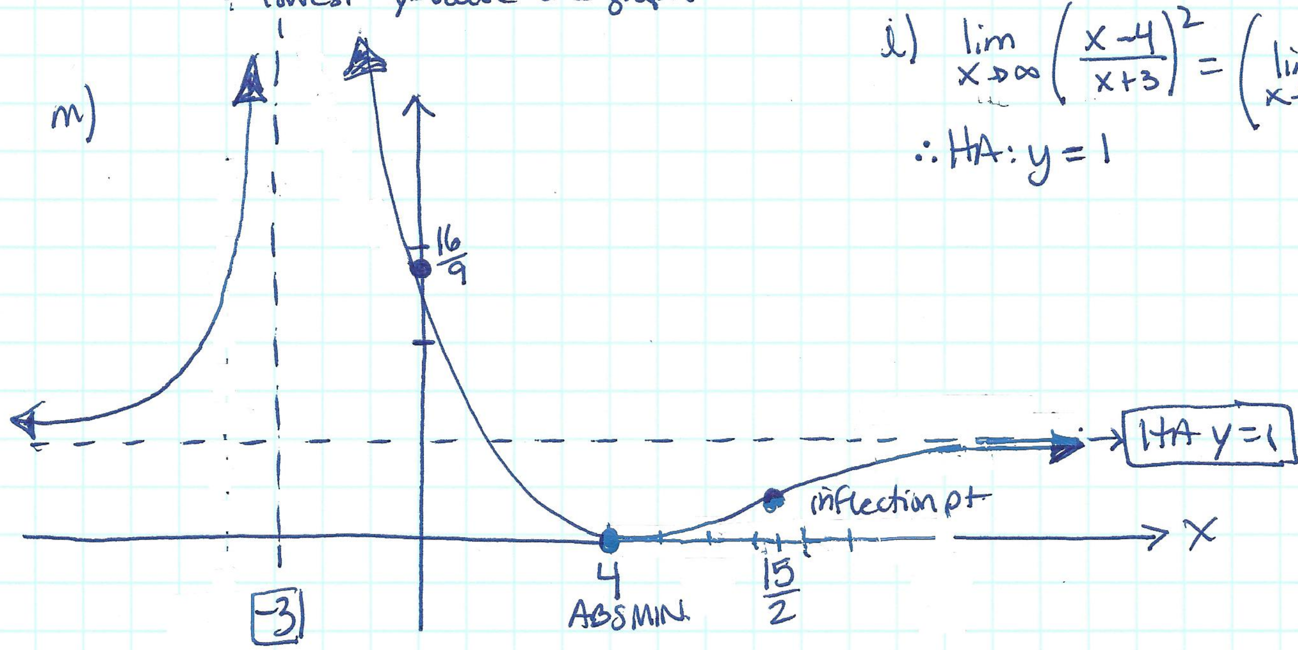
j) Inflection Point @
$\left(\frac{15}{2}, f\left(\frac{15}{2}\right)\right) = \left(\frac{15}{2}, \frac{1}{9}\right)$

k) VA $x = -3$ $\quad$ y-int: $\left(0, \frac{16}{9}\right)$

l) $\lim_{x\to\infty}\left(\frac{x-4}{x+3}\right)^2 = \left(\lim_{x\to\infty}\frac{x-4}{x+3}\right)^2 = 1^2 = 1$

$\therefore$ HA: $y = 1$

m)



UNIT 4 TEST REVIEW.

(17) $f(x) = x^2 e^{-x}$ on $[-1,4]$

a) $f'(x) = 2x e^{-x} - x^2 e^{-x}$
$= e^{-x}(2x - x^2)$ ✓
$= e^{-x}(x)(2-x)$

b) critical pts. $f' = 0$
$x = 0 \quad x = 2$

c) f': $-$ 0 $+$ 0 $-$ on $[-1, 4]$, decreasing on $(-1,0)$, increasing on $(0,2)$, decreasing on $(2,4)$

-1	0	2	4
MAX	MIN	MAX	MIN

d) $f''(x) = (-e^{-x})(2x - x^2) + e^{-x}(2-2x)$
$= e^{-x}(x^2 - 2x - 2x + 2)$
$= e^{-x}(x^2 - 4x + 2)$ ✓
$= e^{-x}((x-2)^2 - 2)$

e) $f'' = 0 \quad x = 2 \pm \sqrt{2}$

f) f'': $+$ 0 $-$ 0 $+$ on $[-1, 4]$

-1	$2-\sqrt{2}$	$2+\sqrt{2}$	4
	0.5857	3.4142	

g) f is increasing on $(0,2)$ b/c $f' > 0$.
decreasing on $(-1,0)\ (2,4)$ b/c $f' < 0$.

h) f is concave up on $(-1, 2-\sqrt{2})\ (2+\sqrt{2}, 4)$ b/c $f'' > 0$.
concave down on $(2-\sqrt{2}, 2+\sqrt{2})$ b/c $f'' < 0$.

** CALCULATOR

i)

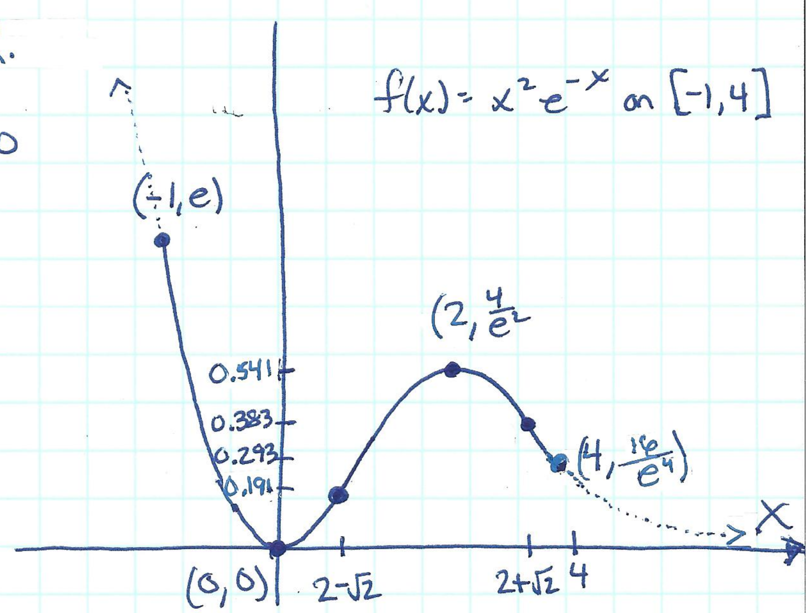
x	-1	0	2	4
y	e	0	$\frac{4}{e^2}$	$\frac{16}{e^4}$
	2.718		0.5413	0.2931
Classify & Justify	ABS MAX	ABS MIN b/c f' changes $\ominus$ to $\oplus$	REL MAX b/c f' changes $\oplus$ to $\ominus$	REL MIN

j) Inflection Points.
$(2-\sqrt{2},\ 0.1910)$
b/c f'' changes signs $\oplus$ to $\ominus$.

$(2+\sqrt{2},\ 0.3835)$
b/c f'' changes signs $\ominus$ to $\oplus$.

k) NOTE: $f(x) = x^2 e^{-x} > 0$ for $x \in \mathbb{R}$.

$\lim_{x \to \infty} (x^2 e^{-x}) = 0 \therefore$ HA $y = 0$



UNIT 4 TEST REVIEW.

18) $f(x) = 2 - x^{\frac{2}{3}}$ $f(x) \geq 0$ for all $x \in \mathbb{R}$.   f

a) $f'(x) = -\frac{2}{3}x^{-1/3}$

$f'(x) = \dfrac{-2}{3x^{1/3}}$

d) $f''(x) = \frac{2}{9}x^{-4/3}$

$f''(x) = \dfrac{2}{9x^{4/3}}$

b) CRIT. PT. $f'(x)$ und: $x = 0$

e) $f''(x)$ und: $x = 0$

c) f' + ∅ −
$\nearrow$ 0 $\searrow$
MAX

f) f'' + 0 +
$\cup$ 0 $\cup$
NO INF. PT.

g) $f(x)$ is increasing on $(-\infty, 0)$ b/c $f' > 0$
decreasing on $(0, \infty)$ b/c $f' < 0$.

h) $f(x)$ is concave up on $(-\infty, 0)(0, \infty)$
& never concave down.

i)

x	0
y	2

ABS MAX
b/c f' changes sign ⊕ to ⊖

j) Inflection Points
none b/c f'' never changes sign.

GRAPH $f(x) \geq 0$ for all $x \in \mathbb{R}$ & $f(x) = 0 = 2 - x^{2/3}$

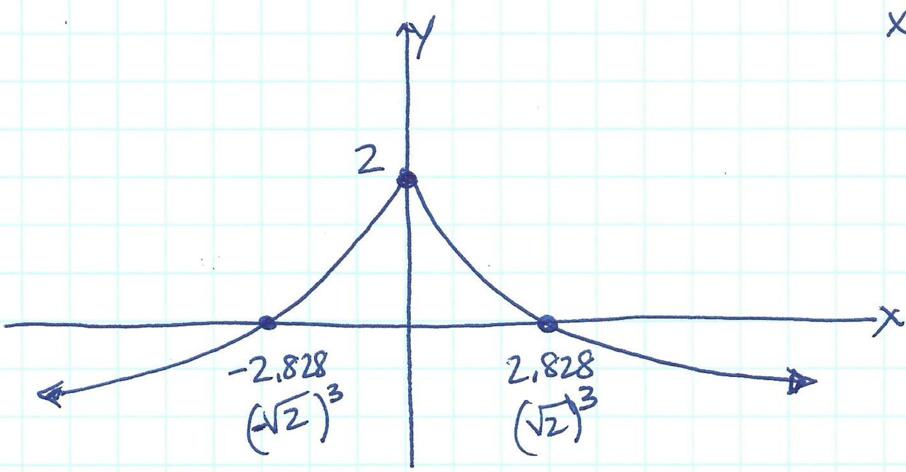
$x^{2/3} = 2$

$x = \pm 2^{3/2}$

zeros **

$x = +2^{3/2} = 2.828$

$x = -2^{3/2} = -2.828$



CHALLENGE:

$$y = x^{1-x}$$

$$\ln(y) = (1-x)\cdot\ln(x)$$

$$\frac{d}{dx}\left(\ln y = (1-x)\ln x \right)$$

$$\frac{1}{y}\frac{dy}{dx} = (-1)\ln x + (1-x)\frac{1}{x}$$

$$\frac{dy}{dx} = \left(\frac{1-x}{x} - \ln x\right) y$$

$$\frac{dy}{dx} = \left(\frac{1-x}{x} - \ln x\right) x^{1-x} \longrightarrow \text{B}$$

$$\frac{dy}{dx} = (1-x)x^{-x} - x^{1-x}(\ln x) \longrightarrow \text{C}$$

$$\frac{dy}{dx} = x^{-x} - x^{1-x} - x^{1-x}(\ln x)$$

$$\frac{dy}{dx} = x^{-x} - (x^{1-x})(1+\ln x) \longrightarrow \text{D}$$