2.1 How Do We Measure Speed?—Student Notes HH6ed

Part I: Using a table of values for a position function

The table below represents the position of an object as a function of time. Use the table to answer the questions that follow.

Time (sec)	Position (m)
2.8	7.84
2.9	8.41
3.0	9.00
3.1	9.61
3.2	10.24
3.3	10.89

1. What is the object's position at time $t = 3$ sec?

   at time $t = 3.3$ sec?

2. What is the *total change* in the object's position over the time interval from 3 to 3.3 sec?

3. Find the *average rate of change* in the object's position over the time interval from 3. to 3.3 sec. Show your work. Include units.

4. By what familiar name do we refer to *average rate of change in position?*

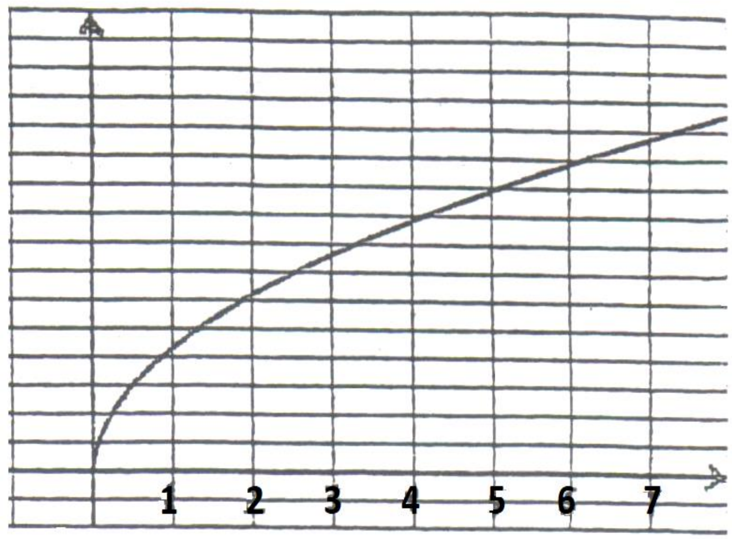
5. Estimate the *instantaneous rate of change* in the object's position at time $t = 3$ sec. Show work. Include units.

6. By what familiar name do we refer to *instantaneous rate of change of position*?

7. Find two other reasonable estimates for the object's velocity at time $t = 3$ sec. Show work.

8. Of your three estimates for velocity at $t = 3$ sec, which one do you prefer? Why?

Part 2: Using a graph of a position function



The graph shown represents the object's position, in miles, as a function of time, in hours since noon. (The vertical scale is intentionally omitted.)

1. Does the object cover a greater distance over the two-hour time interval beginning at noon or over the two-hour time interval beginning at 4:00 p.m.? Explain.

2. Does the object have a greater average velocity over the two-hour time interval beginning at noon or over the two-hour time interval beginning at 4:00 p.m.? Explain

3. Is the object traveling faster at 1:00 p.m. or at 4:00 p.m.? Explain.

Label the scale of the *y*-axis so that 1 block = ½ mile.

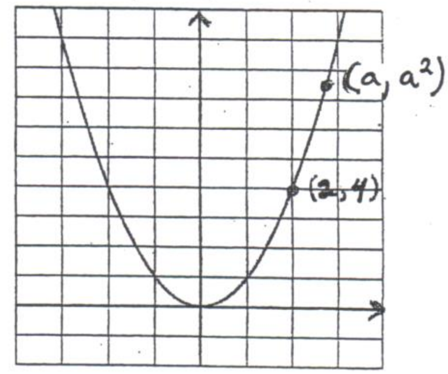
4. Refer back to question #1, but now calculate the distances covered over the two given time intervals. Then decide if your answer to question #1 was correct.

5. Refer back to question #2, and calculate the average velocities over the two time intervals. Draw the secant lines on the graph. Show work (including units) and decide if your answer to #2 was correct.

6. Refer back to question #3 and estimate the instantaneous velocities at the two specified points in time. Draw the tangent lines. Show work (including units) and decide if your answer to #3 was correct.

Part 3: Moving closer to a formal definition of instantaneous velocity

Consider the function $s(t) = t^2$ shown on the graph below. Suppose this function represents the position (meters) of an object at time t (seconds). How can we find the object's instantaneous velocity at a particular point in time, for example at $t = 2$? Is it even possible to do so? On the given function s, the point (2, 4) has been labeled along with a second arbitrary point (a, a^2). Answer the following questions. Many of the answers will be expressions in terms of a.



1. What does the quantity $s(2)$ represent? What is its value?

2. What does the quantity $s(a)$ represent? What is its value?

3. Write an expression for the total change in the object's position on the time interval $[2, a]$.

4. Write an expression for the object's average velocity on the time interval $[2, a]$.

5. We can use the object's average velocity on the interval $[2, a]$ to approximate the object's instantaneous velocity exactly at the time $t = 2$. Of course, as the interval $[2, a]$ gets smaller and smaller (i.e., as the value of a gets closer and closer to 2), the closer the average velocity will approximate the instantaneous velocity.

a. Calculate the average velocity of the object on the following time intervals. Show work.

[2, 2.1]	[1.9, 2]
[2, 2.01]	[1.99, 2]
[2, 2.001]	[1.999. 2]
[2, 2.0001]	[1.9999, 2]

b. Just how small must we make the interval [2, *a*] in order to get the exact value for the instantaneous velocity at $t = 2$? The answer **is infinitely small!** If we find the limit of the average velocity as the time interval [2, *a*] shrinks to zero (i.e., as $(a - 2) \to 0$ or $a \to 2$), we will know the exact value of the instantaneous velocity at time $t = 2$. Use the result of the previous question to estimate the answer. Then, find the limit algebraically, using the expression for average velocity that you wrote in #4.

c. What is the sign of the average velocities on the interval $(-\infty, 0)$? Why?

d. What is the sign of the average velocities on the interval $(0, \infty)$? Why?

So, formally <u>average velocity</u> is the ratio of a change in position (distance) to a change in time. Velocity can be positive, zero or negative, depending on the direction traveled. If two points on the position function $s(t)$ are $(a, s(a))$ and $(b, s(b))$ then

$$\text{Average velocity} = v(t) = \frac{\textit{change in position}}{\textit{change in time}} = \frac{\Delta s}{\Delta t} = \frac{s(b) - s(a)}{b - a}$$

<u>Speed</u> is the magnitude of velocity and is always positive or zero.

When you drive around town you calculate your average speed because you are not concerned about the direction you are traveling, only the distance you are traveling.

<u>Instantaneous velocity</u> refers to the velocity at a particular point in time.

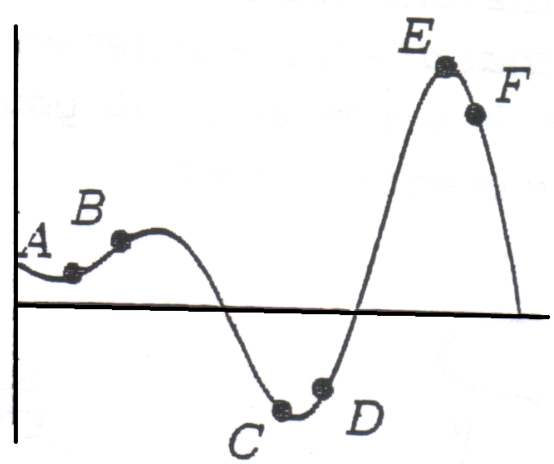
$$\text{Instantaneous velocity} = \lim_{b \to a} \frac{s(b) - s(a)}{b - a}$$

It is also the slope of the tangent line to the curve at that point.

<u>Part 4: How Do We Measure Speed? How About Velocity?</u>

1. Match the points labeled on the curve with the given slopes.

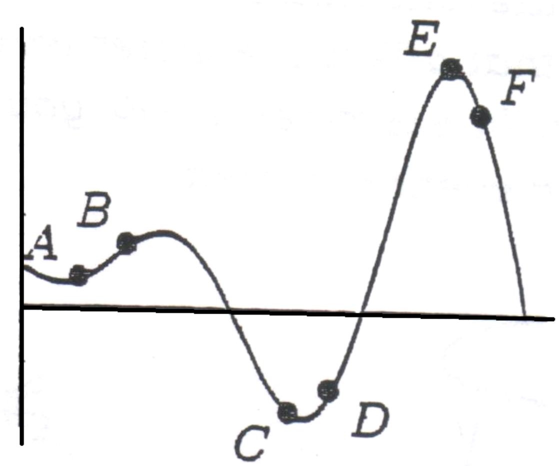
Slope	Point
-3	
-1	
0	
½	
1	
2	



2. For the graph, arrange the following numbers in ascending order.

_____the slope of the graph at A
_____the slope of the graph at E
_____the slope of the line EF
_____the slope of the line AB
_____the slope of the graph at C

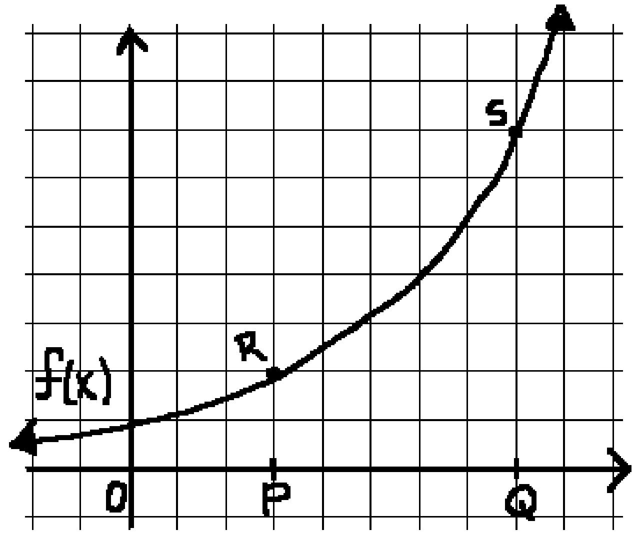
3. Suppose that the graph is of the velocity vs. time of a butterfly in flight. When the velocity is positive, the butterfly is flying upward. When the velocity is negative, the butterfly is flying downward. Match the labeled point(s) with the appropriate description. Some descriptions may fit more than one point.



_____the butterfly is flying the fastest
_____the butterfly's velocity is increasing but at a decreasing rate
_____the butterfly is in a dive toward a tasty flower
_____the butterfly is flying the slowest
_____the butterfly's velocity is decreasing
_____the butterfly's speed is decreasing

4. Find a point on the graph where the butterfly changes direction. Label it G. There may be more than one.

2.2 The Derivative at a Point--Student Notes HH6ed



Refer to the graph at the right of some arbitrary function *f*.

1. Let *a* represent the distance from the origin *O* to the point *P*. Label it on the graph.
Identify coordinate P

P(_____,_____)

2. Let *h* represent the distance from point *P* to point *Q*. Label it on the graph. Outline it in blue
Identify coordinate Q.

Q(_____,_____)

3-4. Outline segments $\overline{RP}$ *and* $\overline{SQ}$ in green.
Write the algebraic expressions for the lengths of $\overline{RP}$ *and* $\overline{SQ}$ and identify coordinate R and S.

RP = __________ *SQ* = __________ *R*(_______,_______) *S*(________,__________)

5. On the figure draw and label the segment whose length is $f(a+h)-f(a)$ in blue.

6. Draw the secant line *RS* in blue. Write an algebraic expression for its slope.
Simplify completely.

7. Suppose you were to take the limit of the slope expression you just wrote as *h* gets infinitely small.
What would this limit represent geometrically?

8. Sketch the tangent line to the function *f* at the point *R* in red.

9. Write an algebraic expression for the slope of this line
(Hint: Recall the relationship between average velocity and instantaneous velocity.)

10. What notation do we use for this quantity?

11. What special name do we reserve for this quantity?

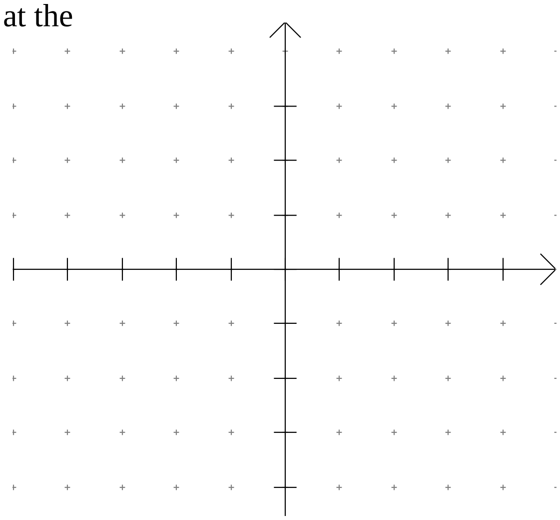
Conclusion: The instantaneous rate of change or the derivative is $f\,'(a)=\lim\limits_{h\to 0}\dfrac{f(a+h)-f(a)}{h}$

Conclusion: The instantaneous rate of change or the derivative is $f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$

Practice: For each function, make a sketch of the curve and use your straight edge to draw the tangent line to the curve at the give point.

a. Estimate the slope of the curve at the point using your tangent line (show work)
b. Find the actual slope of the curve at the point using the definition of derivative
c. Write the equation of the tangent line to the curve at the point.

12. $f(x) = x^2 + 1$ at $x = 1$

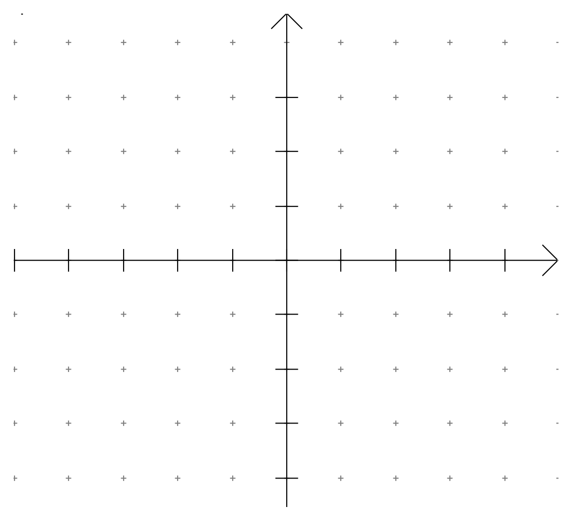


a.

b.

c.

13. $f(x) = \frac{1}{x}$ at $x = 1$



a.

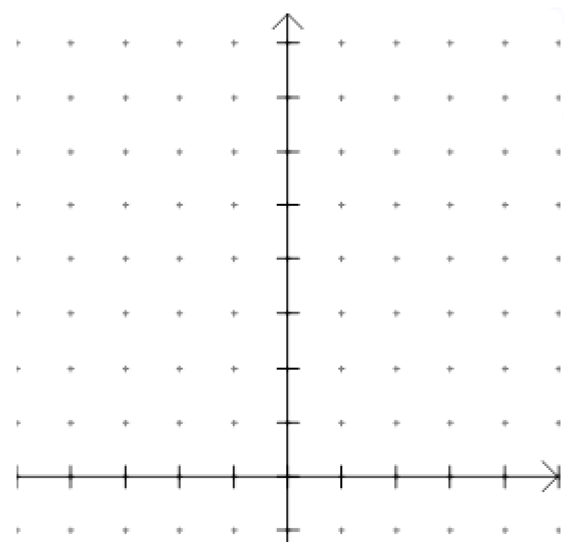
b.

c.

14. Find the derivative of $f(x) = 5x^2$ at $x = 10$ using the definition of derivative.

15. Find the equation of the line tangent to the function $f(x) = x^3$ at $x = -2$ using the definition of derivative.

16. a. Sketch the graphs of the functions
$f(x) = \frac{1}{2}x^2$ *and* $g(x) = f(x) + 3$ on the same set of axes.

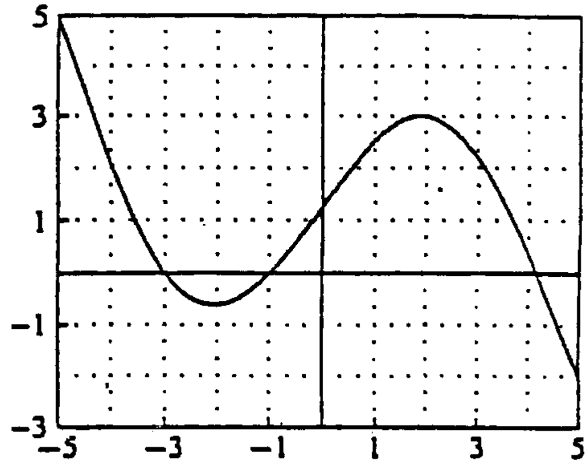


b. What can you say about the slopes of the tangent lines to the two graphs at the point $x = 0$? $x = 2$? $x =$ any c?

c. Explain why adding a constant value, c, to any function does not change the value of the slope of its graph at any point.

<u>Review of Terminology</u>: Refer to $f(x)$ with domain $[-5,5]$ to answer the following questions.

1. Specify the intervals on which



Graph of *f*

a. *f* is positive
f. *f* is negative

b. *f* ’ is positive
g. *f* ’ is negative

c. *f* is increasing
h. *f* is decreasing

d. *f* ’ is increasing
i. *f* ’ is decreasing

e. *f* is concave up
j. *f* is concave down

2. Complete the following statements using the problem above:

a. If *f* ’ > 0, then *f* is _______________
b. If *f* ’ < 0, then *f* is _______________

c. If *f* ’ is increasing then *f* is ________
d. If *f* ’ is decreasing then *f* is ________

3. a. What is the maximum value of *f*?____________ Where does it occur?______________

b. Estimate the maximum slope of *f* ’. Explain. At what *x*-value does this occur?

c. What is the maximum value of *f* on the interval [0, 4]?____________________________

Where does this max occur?________________What is the value of *f* ’ at this point?____

d. What is the minimum value of *f* on the interval [-5, 0]?____________________________

Where does this min occur?________________What is the value of *f* ’ at this point?____

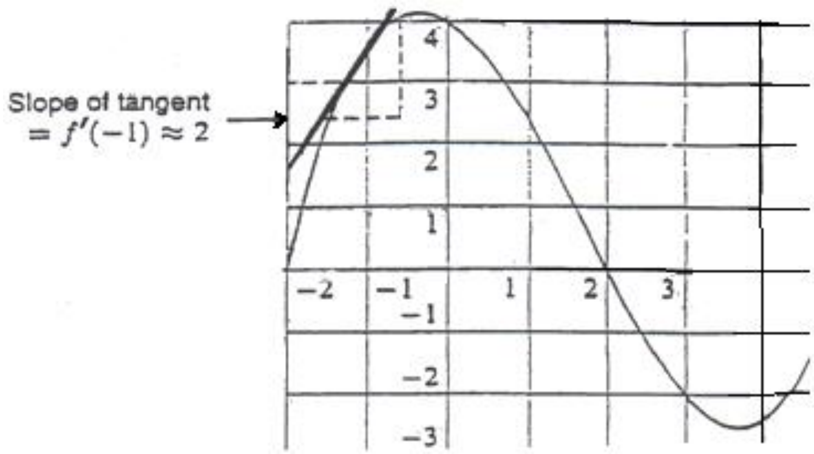
4. In each of the given pairs of expressions, determine which is larger. Explain.

a. *f*(2) or *f*(3)
b. *f* ’(2) or *f* ’(3)

c. *f*(1) – *f*(0) or *f*(2) – *f*(1)
d. $\dfrac{f(1)-f(0)}{1-0}$ *or* $\dfrac{f(2)-f(0)}{2-0}$

2.3 The Derivative Function--Student Notes HH6ed

The graph at the right is some function $f(x)$ with a tangent drawn to the curve at the point $x = -1$.



1. Complete the table below by indicating, for each of the specified intervals, whether the **slope** of the function is positive or negative, increasing or decreasing on the interval.

Interval	Positive/Negative	Increasing/decreasing
(-2, -0.5)		
(-0.5, 1.5)		
(1.5, 3.5)		
(3.5, 5)		

2. Using a straightedge and pencil, lightly sketch the tangents to the function and estimate the slopes of the tangents. Complete the table of values for the derivative function below.

x	-2	-1	0	1	2	3	4	5
$f'(x)$								

3. Using a colored pencil, sketch the graph of the derivative function by plotting your table values and connecting them with a smooth curve. Do this on the grid of $f(x)$.

4. Based on the graphs of $f(x)$ and its derivative $f'(x)$, answer these questions:

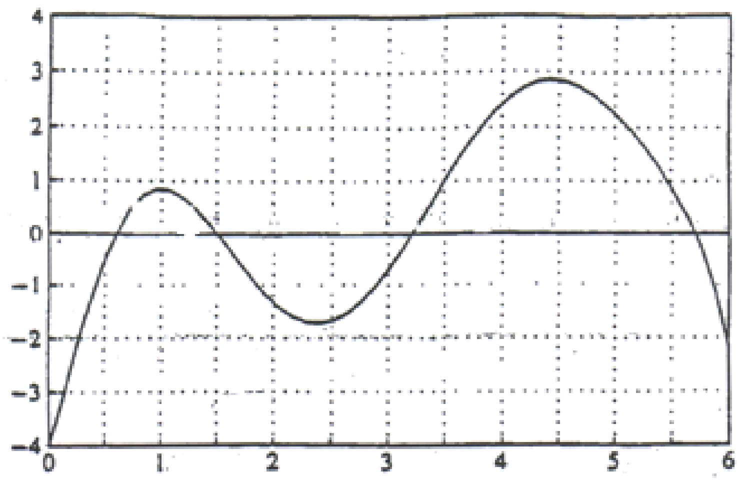
   a. When the derivative function $f'(x)$ is positive, the graph of $f(x)$ is __________________________

   b. When the derivative function $f'(x)$ is negative, the graph of $f(x)$ is _________________________

   c. When the derivative function $f'(x)$ changes sign, the graph of $f(x)$ is ________________________

   d. When the derivative function $f'(x)$ has a turning point, the graph of $f(x)$ is ____________________

5. The graph of the function f is shown below.

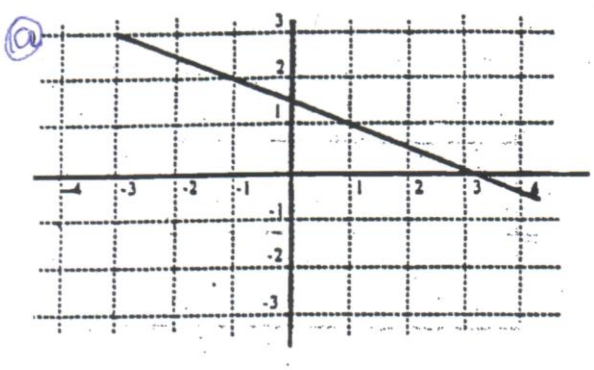


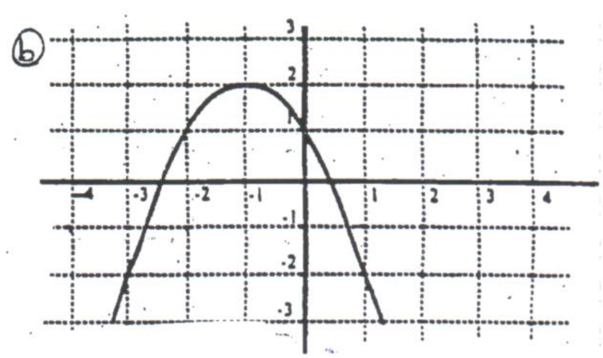
a. Complete the table below, filling in the values for $f\,'(x)$.

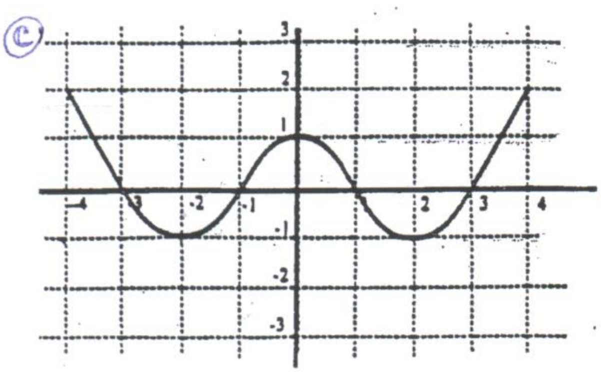
x	1	2	3	4	5	6
$f'(x)$						

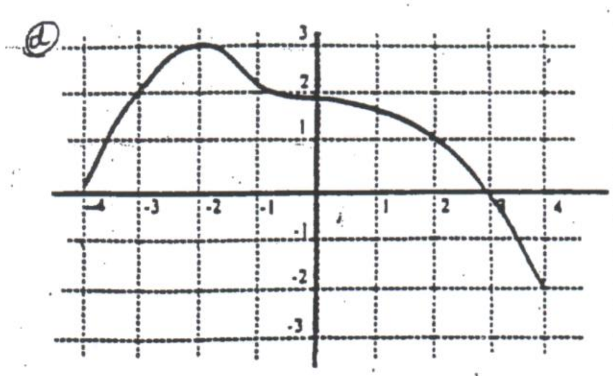
b. Sketch a graph of $f\,'(x)$. Do this on the grid of $f(x)$.

6. For each of the following, sketch a graph of the derivative function on the axes with the function. Use a colored pencil.





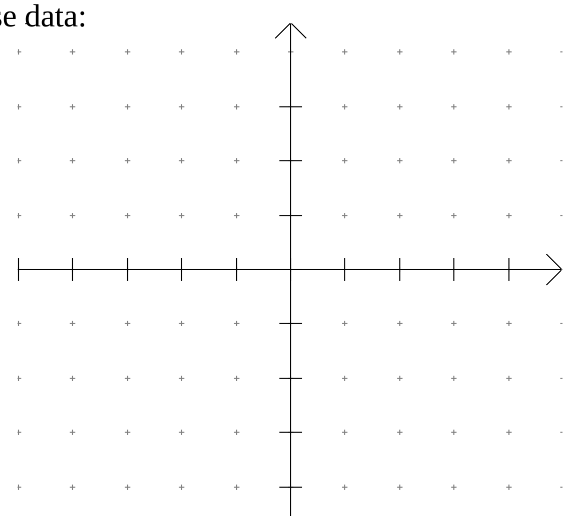




Practice:

1. a. Sketch the graph of a function f that is consistent with these data:

x	-2	-1	1	2
$f(x)$	1	-1	-1	2
$f'(x)$	-3	0	-1	-2



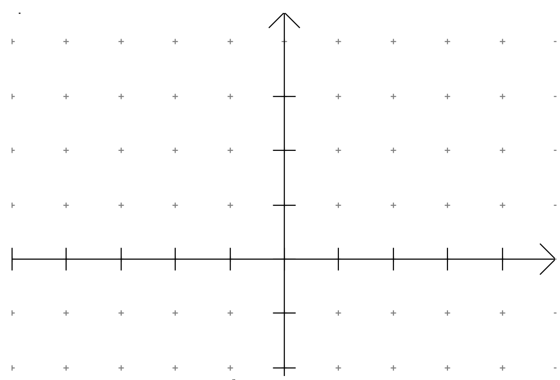
b. Write an equation for the tangent line to the function f at $x = -2$.

2. The line tangent to a function f at (5, 2) passes through the point (0, 1). Find $f(5)$ and $f'(5)$.

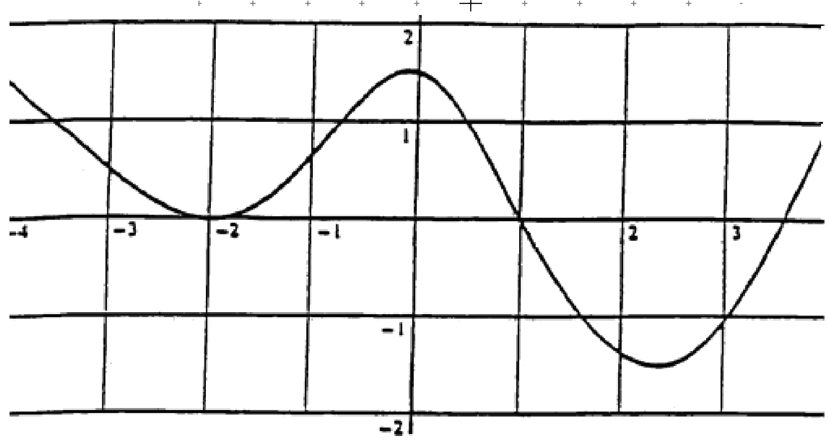
3. Suppose that $f'(x) \geq 0$ on the interval (2, 7). Explain why $f(3) \leq f(6)$.

4. Draw the continuous function $y = f(x)$ that satisfies the following three conditions.

- $f'(x) > 0$ for $x < -2$
- $f'(x) < 0$ for $-2 < x < 2$
- $f'(x) = 0$ for $x > 2$

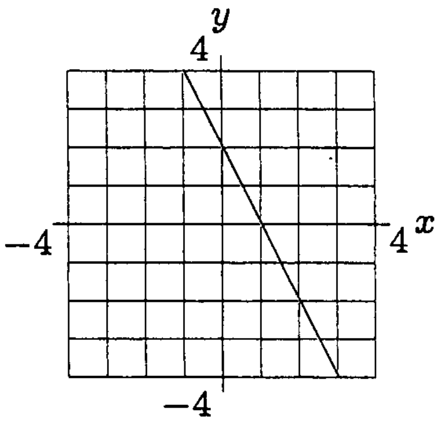


5. The graph f is given.
Sketch the graph of f'.



6. For exercises 1-8, sketch a graph of the derivative function of each of the given functions.

1.



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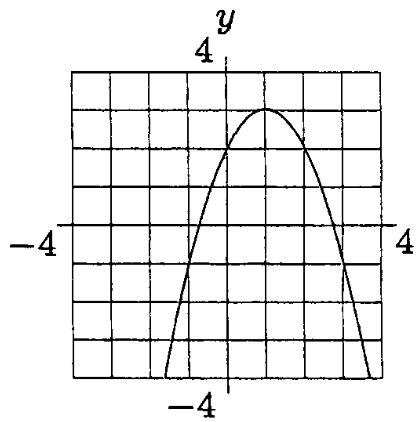
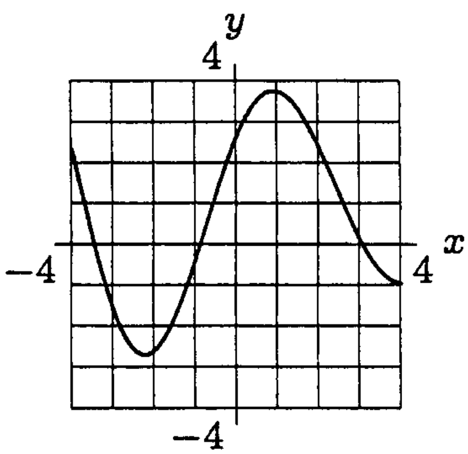
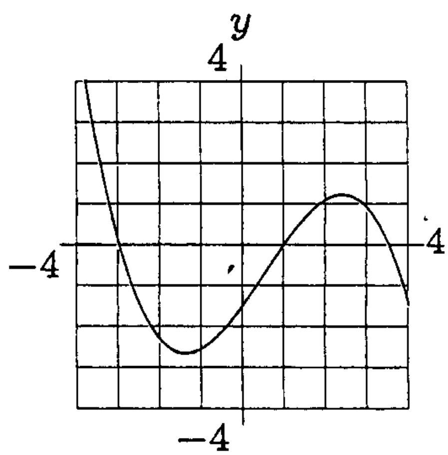


5.



6.



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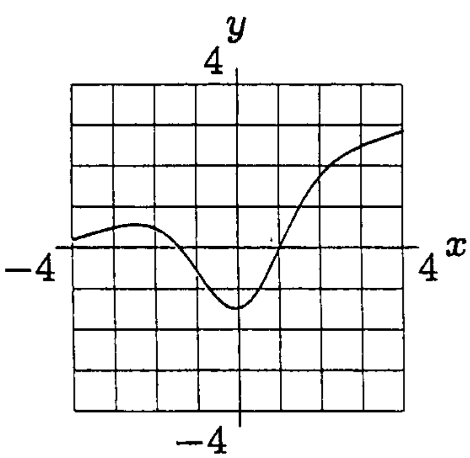

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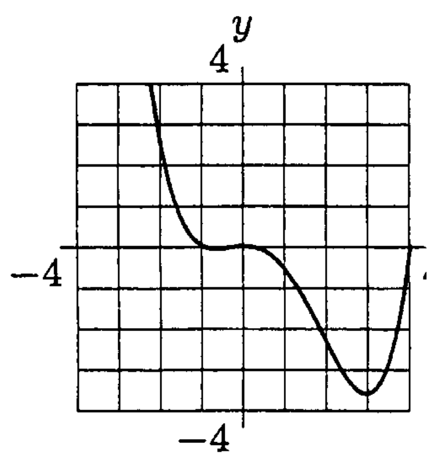


2.4 Interpretations of the Derivative—Student Notes HH6ed

An Alternative Notation for the Derivative and Interpreting its Meaning

Given that $y = f(x)$, the derivative can be written as $f'(x)$ or $\frac{dy}{dx}$.

The second notation was introduced by Wilhelm Gottfried Leibniz, a German mathematician. The letter d stands for "small difference in . . ." so literally the notation $\frac{dy}{dx}$ can be thought of as

$$\frac{\text{Small difference in } y\text{-values}}{\text{Small difference in } x\text{-values}}$$

We say "the derivative of y with respect to x."

Example 1: Use the definition of derivative to find a formula for $\frac{dy}{dx}$ algebraically given $f(x) = x^2 - x$.

If we want to indicate that you should find the derivative at $x = 2$, you write $f'(2)$ or $\left.\frac{dy}{dx}\right|_{x=2}$

Example 2: Suppose $s = f(t)$ gives the distance, in meters, of a body from a fixed point as a function of time t, in seconds.

a. Describe the following in real-world terms: $\left.\frac{ds}{dt}\right|_{t=2}$

b. What are the units associated with this quantity?

c. What is the common term for $\frac{ds}{dt}$?

d. What is the real-world meaning of $f'(2) = 10$? Use units in your answer.

Example 3: The cost, C, in dollars, of building a house A ft^2 in area is given by the function $C = f(A)$.

a. What is the real world meaning of $f(2000) = 195{,}000$? Use units in your answer.

b. What is in the independent variable? Dependent variable?

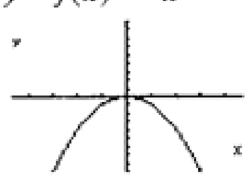
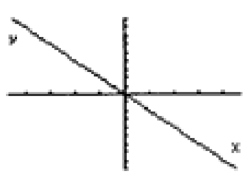
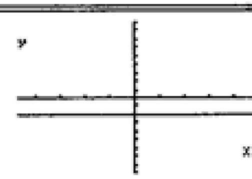
c. What is the sign of $f'(A)$? Why?

d. Rewrite $f'(A)$ in Leibniz's notation.

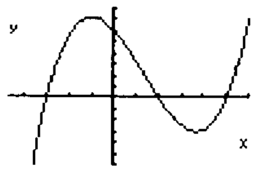
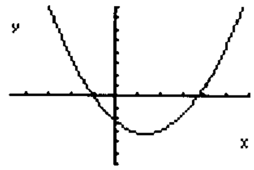
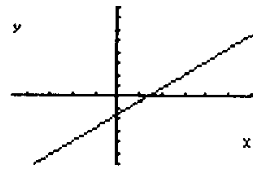
e. What are the units of $f'(2000)$?

f. What is the real world meaning of $f'(2000) = 150$?

Analyze the graph of $y = -x^2$, using the first and second derivative graphs.

Graph of f(x)	***Graph of f '(x)***	***The value of the slope of tangent line***	***Graph of f ''(x)***	***Observations***
$y = f(x) = -x^2$  • *for x <0 f is increasing* • *for x > 0 f is decreasing* • *f is always concave down*	 • *for x < 0 f '(x) is positive.* • *for x > 0 f '(x) is negative.* • *f '(x) is always decreasing*	*f '(-2) = 4* *f '(-1) = 2* *f '(0) = 0* *f '(1) = -2* *f'(2) = -4* *f '(x) is always decreasing*	 *f ''(x) < 0 for all x.*	On the same interval: • *f(x) is concave down* • *f '(x) is decreasing* • *f ''(x) < 0*

Observe the relationship between the graphs of *f(x), f '(x) and f ''(x).*

Graph of f(x)	***Graph of f'(x)***	***Graph of f''(x)***	***Observations***
 *f(x) changes concavity*	 *f '(x) decreases then increases*	 *f ''(x) is negative then positive*	• *Where f ''(x) changes from negative to positive, f(x) changes concavity.* • *Where f ''(x) = 0, f(x) changes concavity.* • *Where f '(x) changes from decreasing to increasing, f ''(x) = 0.*

<u>Conclusions:</u>

<u>The First Derivative:</u>

- If $f'(x) > 0$ on an interval, then $f(x)$ is ________________________ over that interval.
- If $f'(x) < 0$ on an interval, then $f(x)$ is ________________________ over that interval.
- If $f'(x) = 0$ on an interval, then $f(x)$ has a ________________________ at *x* which is

  either a ________________________ when ___________ changes sign

  or a ___________________________ when ____________ does not change sign.

<u>The Second Derivative:</u>

- If $f''(x) > 0$ on an interval, then $f'(x)$ is _________________________ and

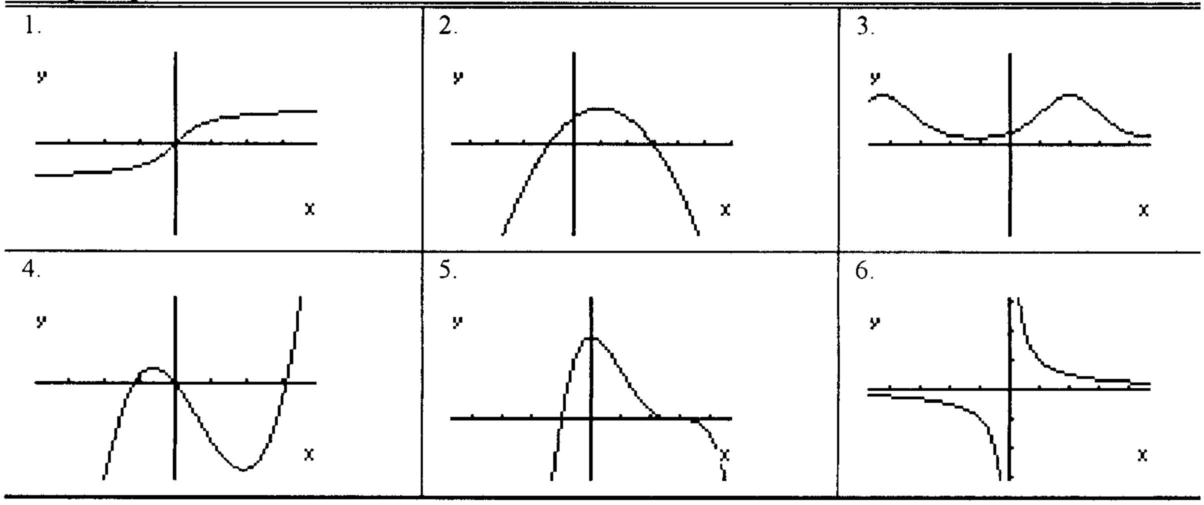
  $f(x)$ is __________________________ over that interval.
- If $f''(x) < 0$ on an interval, then $f'(x)$ is _________________________ and

  $f(x)$ is __________________________ over that interval.
- If $f''(x) = 0$ on an interval, then $f(x)$ sometimes has a ________________________________ at

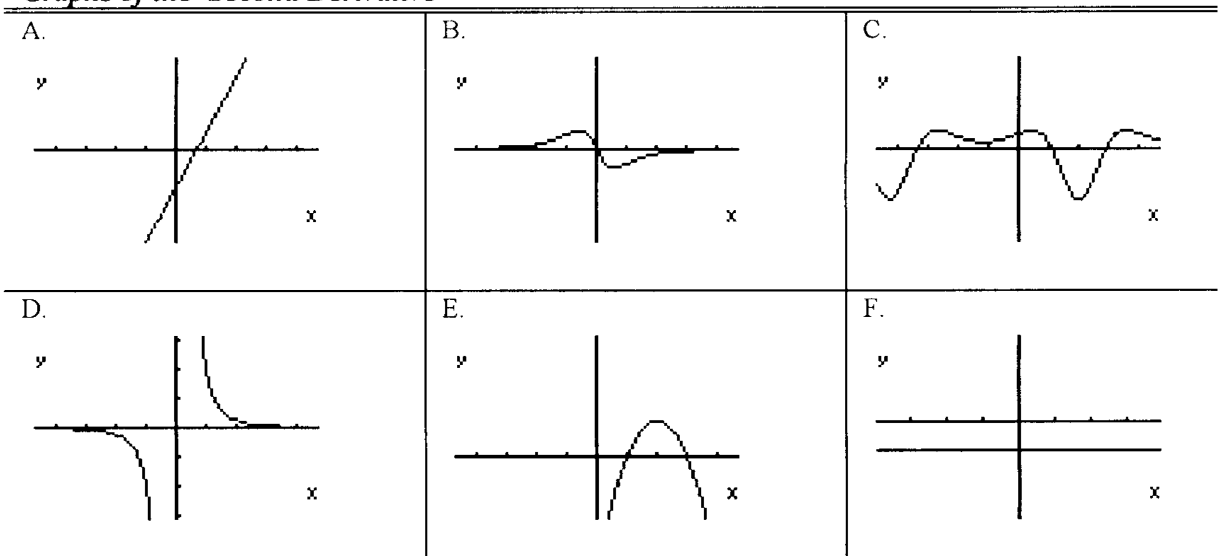
  that value of x, but only if ______________ changes sign.

Match the graph of each function below 1 - 6, with the graph of its second derivative A - F.

Graphs of the Function



Graphs of the Second Derivative

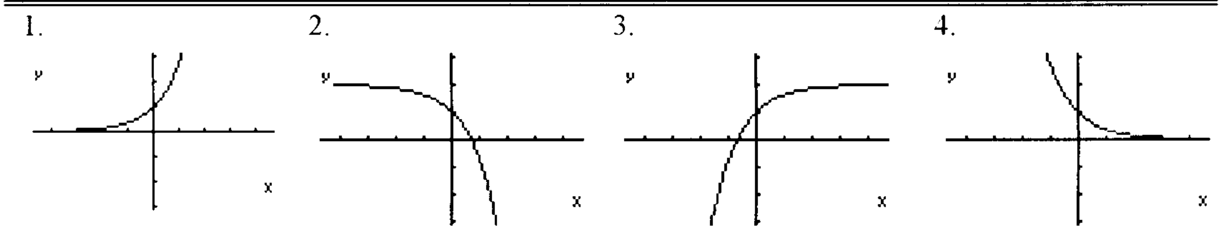


Determine which of the functions graphed below is

a) increasing at an increasing rate,
b) increasing at a decreasing rate,
c) decreasing at an increasing rate or
d) decreasing at a decreasing rate.

Explain why you have chosen each.

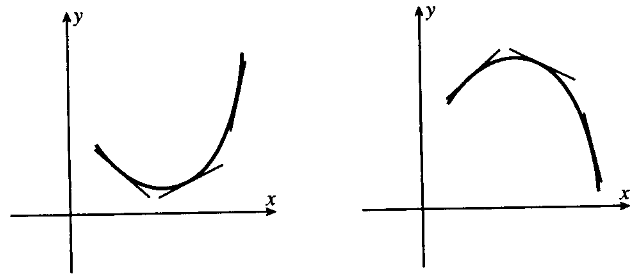
Graphs of Increasing and Decreasing functions



2.5 The Second Derivative Function—Student Notes HH6ed

A function $f(x)$ is ***concave upward*** on an interval I if $f(x)$ lies above all tangent lines to $f(x)$ in I.

A function $f(x)$ is ***concave downward*** on an interval I if $f(x)$ lies below all tangent lines to $f(x)$ in I.

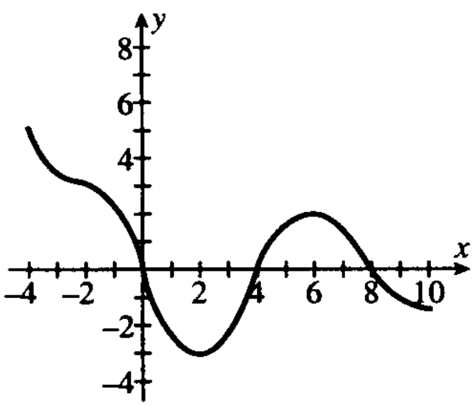


concave upward   concave downward

The test for concavity involves the second derivative: If $f(x)$ is twice differentiable on an interval I (meaning $f''(x)$ exists for all x on the interval I) then

a. If $f''(x) > 0$ for all x on the interval I, then f is *concave upward* on I.
b. If $f''(x) < 0$ for all x on the interval I, then f is *concave downward* on I.

Example 1: Use the graph below to answer true or false to each.



a) $f''(x) > 0$ for $x \in (2, 4)$

b) $f''(x) < 0$ for $x \in (-4, -2)$

c) $f''(6) = 0$

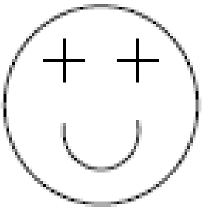
d) $f''(2) > 0$

e) f is concave upward on $(0, 2)$

The concavity test can be remembered with the following pictures … keep in mind these are NOT to be used for justification.

f'' positive

f Concave UP



f'' negative

f Concave DOWN

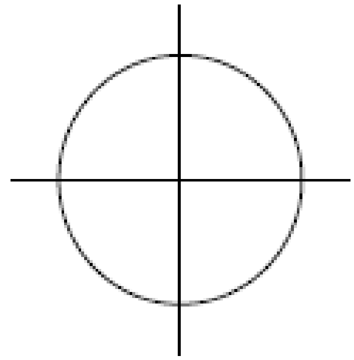
Example: Label each quadrant below with one of the following descriptions:

i) Increasing and Concave Up

ii) Increasing and Concave Down

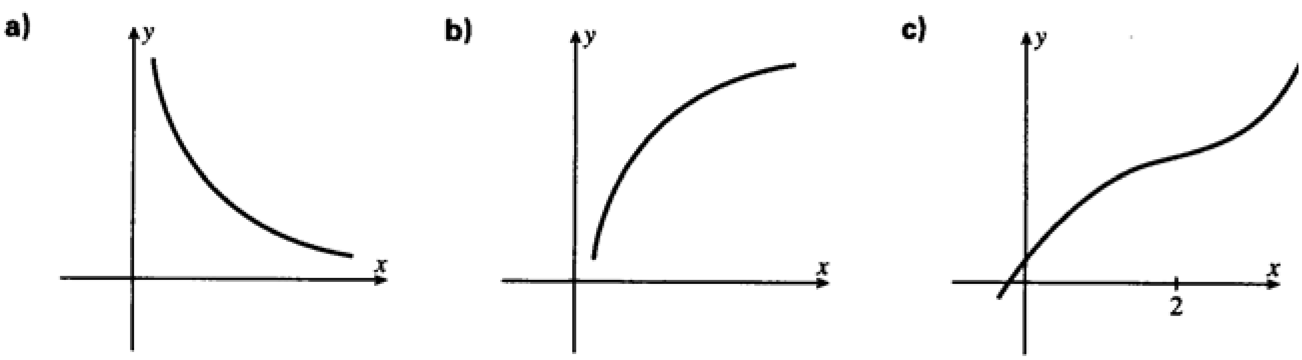
iii) Decreasing and Concave Up

iv) Decreasing and Concave Down.

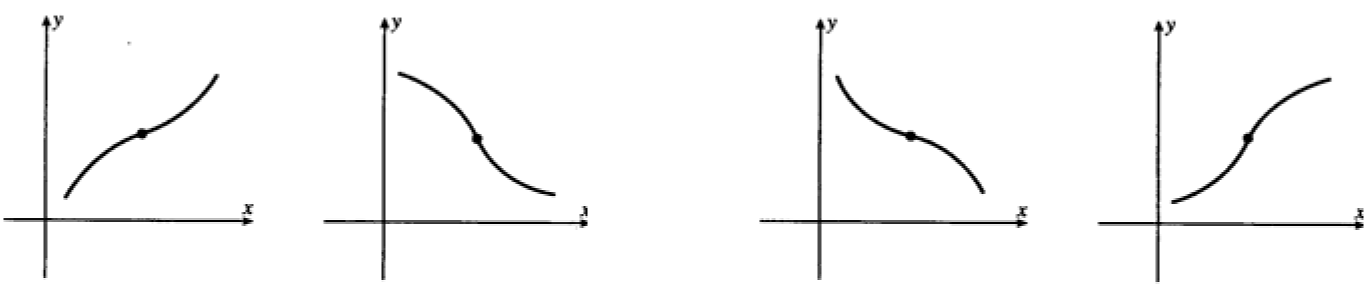


Points of Inflection
A point of inflection is a point on the graph where the concavity changes.

Example 2: The graph of a function f is given. What can be said about $f'(x)$ and $f''(x)$ for each (i.e., positive/negative/where)?



A *point of inflection* for f is a point on the graph of f where concavity changes from concave downward to concave upward or from concave upward to concave downward.

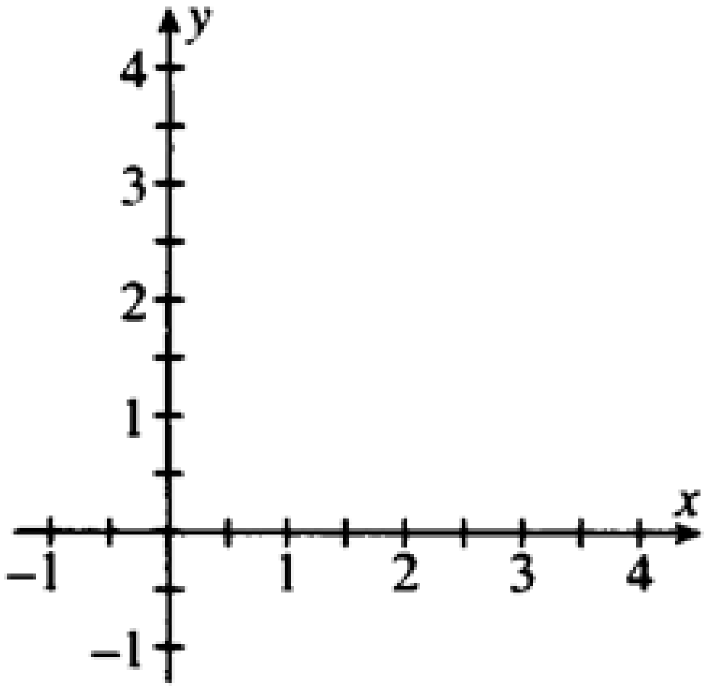


Concave downward to concave upward

Concave upward to concave downward

Example 3: Sketch a graph of a function having all of the following properties.

$f(-1) = 4, f(0) = 2, f(2) = 1, f(3) = 0$
$f'(x) \leq 0$ for $x < 3$ and
$f'(x) \geq 0$ for $x > 3$.
$f''(x) < 0$ for $0 < x < 2$ and
$f''(x) \geq 0$ elsewhere.



2.6 Continuity and Differentiability—Student Notes HH6ed

Definition: A function $f(x)$ is **continuous at a number *a*** if

$$\lim_{x \to a} f(x) = f(a)$$

This definition implicitly requires three things to be continuous at $x = a$:

1. $f(a)$ exists (that is, a is in the domain of $f(x)$)
2. $\lim_{x \to a} f(x)$ exists (so $f(x)$ must be defined on an open interval that contains a)
3. $\lim_{x \to a} f(x) = f(a)$

There are 3 types of Discontinuity:

1. Removable Discontinuity: A limit exists, but there is a hole at the value.
2. Non-removable (or Jump) Discontinuity : A limit does not exist at the value.
3. Infinite Discontinuity: There is a vertical asymptote at the value. The limit from the left and right is ∞ or $-\infty$

Examples:

1. Where are each of the following functions discontinuous? State the type of discontinuity.

a. $f(x) = \dfrac{x^2 - x - 2}{x - 2}$

b. $f(x) = \begin{cases} \dfrac{1}{x^2}, & \text{if } x \neq 0 \\ 1, & \text{if } x = 0 \end{cases}$

c. $f(x) = \begin{cases} \dfrac{x^2 - x - 2}{x - 2}, & \text{if } x \neq 2 \\ 1, & \text{if } x = 2 \end{cases}$

d. $f(x) = x$

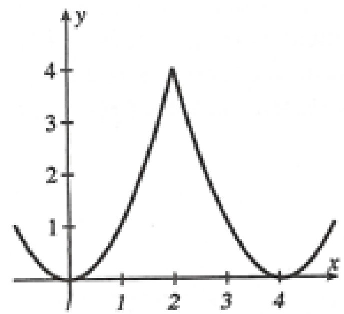
<u>Definition</u> A function $f\left(x\right)$ is **differentiable at *a*** if $f'(a)$ exists. It is **differentiable on an open interval** $\left(a,\, b\right)$ [or $\left(a,\infty\right)$ or $\left(-\infty,a\right)$ or $\left(-\infty,\infty\right)$] if it is differentiable at every number in the interval.

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

There are 3 common ways for a function to fail to be differentiable at a point

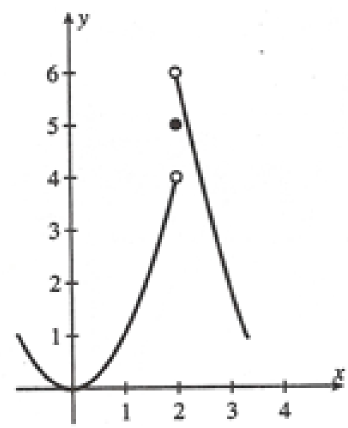
a. The graph has a **sharp point or cusp**.

<u>Example:</u> $f(x)=\begin{cases} x^2 \ \ if \ \ x\le 2 \\ (x-2)^2 \ \ if \ \ x>2 \end{cases}$



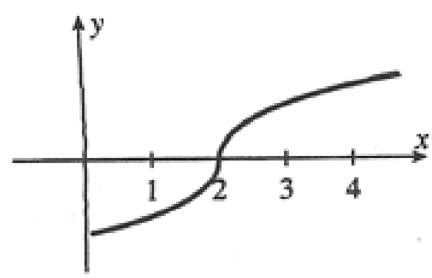
b. The function is **discontinuous**. (break, hole or asymptote)

<u>Example:</u> $f(x)=\begin{cases} x^2 \ \ if \ \ x<2 \\ 5 \ \ if \ \ x=2 \\ 10-x^2 \ \ if \ \ x>2 \end{cases}$



c. The graph has a **vertical tangent line**.

<u>Example:</u> $f(x)=\sqrt[3]{x-2}$



Theorem: If $f\left(x\right)$ is differentiable at a point $x=c$, then $f\left(x\right)$ is continuous at c. The converse is **false**.

* differentiability$\Rightarrow$continuity $\Rightarrow$limit

Examples:

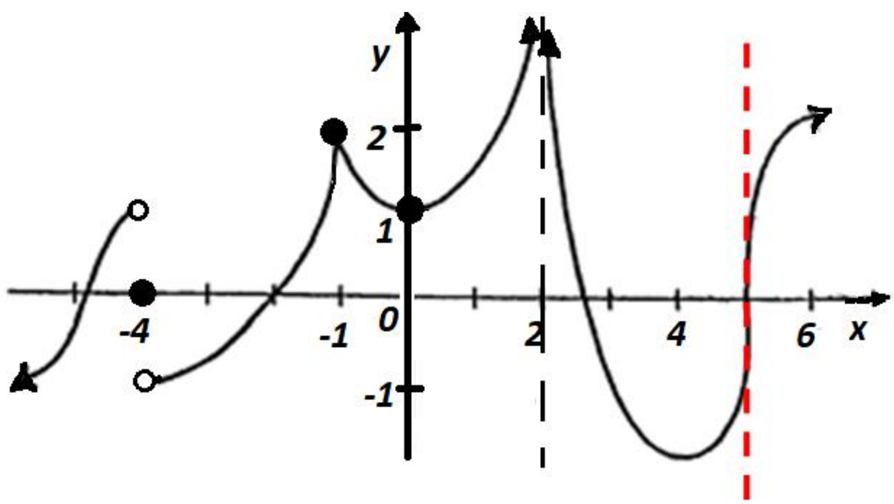
2. Is the absolute value function differentiable at $x=0$? Explain.

3. Is $f(x)=x^{\frac{1}{3}}$ differentiable at $x=0$? Explain.

4. Is $f(x)=(x-1)^{\frac{2}{3}}$ differentiable at $x=1$? Explain.

5. Is $f(x)=\dfrac{x^2-5x+6}{x-3}$ differentiable at $x=3$?

6. Refer to the figure at the right. Complete the following table indicating at which values on the open interval (-6, 6), the given function, *f*, fails to be continuous and/or differentiable.



Domain value	Continuous? (yes or no)	If no, why?	Differentiable? (yes or no)	If no, why
1. $x=-4$				
2. $x=-1$				
3. $x=0$				
4. $x=2$				
5. $x=5$				