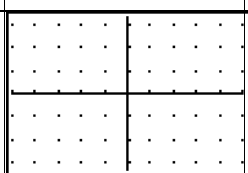
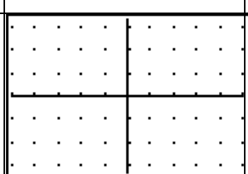
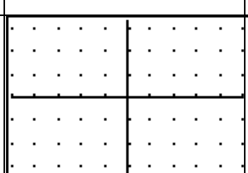
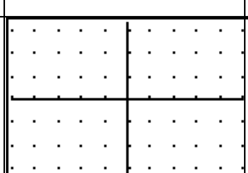
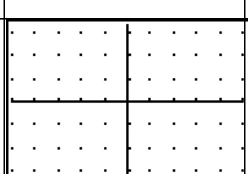
1.7 Continuous Functions – Student Notes

<u>Objective:</u> To understand a continuous function by investigating the relationship between an intuitive sense of continuity and its mathematical definition.

What is Continuity?

1. Which of the following situations could be classified as examples of continuous quantities?
   - Electricity used in a house during one month.
   - The amount of change in your pocket over one school day.
   - The height of the Ocean City tides over one year.
   - The temperature inside your oven during Thanksgiving Day.

2. Use **ZOOM 4: Decimal** window & the table on your calculator to investigate the different types of behaviors near and at $x = 0$. Consider the domain of the function as you investigate the behavior.

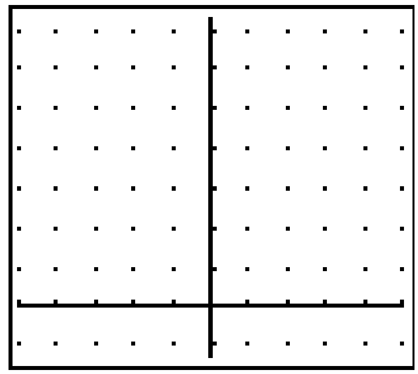
$y = f(x)$	Graph	Find $f(0)$.	Find $\lim_{x\to 0} f(x) =$	Is $f(x)$ continuous at $x = 0$?
$f(x) = x$		$f(0) = 0$	$\lim_{x\to 0} f(x) = 0$	$f(x)$ is continous at $x = 0$ *because* $f(x)$ has no holes, asymptotes, or jumps.
$f(x) = \dfrac{x^2}{x}$				
$f(x) = \dfrac{1}{x}$				
$f(x) = \dfrac{\lvert x\rvert}{x}$				
$f(x) = x^{\frac{2}{3}}$				
$f(x) = \dfrac{\sin(x)}{x}$				

3. Write a definition of continuity at a point $x = c$ in terms of limits and the definition of the function at the point where $x = c$.

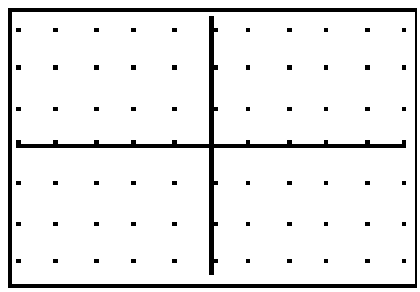
Mathematical Definition of CONTINUITY	Describe 3 types of DISCONTINUITY?
1.____	1.__________________________________
2.____	2.__________________________________
3.____	3.__________________________________

4. Sketch the graph of $f(x) = \begin{cases} 2x+4, & x \in (-\infty, -1] \\ x^2, & x \in (-1, \infty) \end{cases}$.

Is $f(x)$ continuous at $x = -1$? Explain.



5. Sketch the graph of $f(x) = \begin{cases} -1, & x \in (-\infty, -2] \\ 2, & x \in (-2, 0) \\ x, & x \in [0, \infty) \end{cases}$.



Use the graph of $f(x)$ to determine continuity at:

a) $x = -2$? Explain.	b) $x = 0$? Explain.	c) $x = 1$? Explain.

6. Sketch the graph of a function *f* that satisfies all of the following conditions:

- The domain of *f* is $x \in [-3, 3]$.
- $f(-3) = f(-1) = f(1) = f(3) = 2$
- It is discontinuous at -1 and 1.
- *f* is continuous on the open interval $x \in (-1, 1)$.

7. Let $f(x)=\begin{cases} ax^2+2 & if \quad x<-1 \\ x & if \quad x\geq -1 \end{cases}$ What value of a makes f continuous at $x=-1$?

8. Suppose that the function f is continuous at and that f is defined by the rule

$$f(x)=\begin{cases} kx^2+2 & if \quad x<5 \\ 4x+7 & if \quad x\geq 5 \end{cases}$$

a. Find k.
b. *Find $\lim\limits_{x\to 5} f(x)$.

*This means find the y-value that the function gets close to as x-value gets closer and closer to $x=5$.

9. Given two functions $f(x)$ and $h(x)$ such that $f(x)=x^3-3x^2-4x+12$ and $h(x)=\begin{cases} \dfrac{f(x)}{x-3} & for \quad x\neq 3 \\ p & for \quad x=3 \end{cases}$

a. Find all zeros of the function *f*.
b. Find the value of *p* so that the function *h* is continuous at *x* = 3. Justify your answer.
c. Using the value of *p* found in (b), determine whether *h* is an even function. Justify your answer.

(This problem is from the 1976 AP AB Calculus Exam)

10. Given the function f defined by $f(x)=\dfrac{2x-2}{x^2+x-2}$

a. For what values of x is f(x) discontinuous?
b. At each point of discontinuity found in part (a), determine whether f(x) has a limit and, if so, give the value of the limit.
c. Write an equation for each vertical and horizontal asymptote. Justify your answer.
d. Draw a detailed and complete graph of f(x) using the information you found in parts a, b, & c.

(This problem is from the 1978 AP AB Calculus Exam)

1.7.1 NOTES: Limit Properties and Algebraic Techniques to evaluate Limits -- Finding Limits Analytically

Evaluate each of the following limits.

1. $\lim_{x\to 3} 5 =$ ______ 2. $\lim_{x\to 3} x =$ ______ 3. $\lim_{x\to 0}(-6) =$ ______ 4. $\lim_{x\to -0.5} x =$ ______

Use your results to complete the following:

5. For the constant function: $f(x) = A,\ \lim_{x\to c} f(x) = \lim_{x\to c} A =$ ______ , where c is any number.

6. For the identity function: $f(x) = x,\ \lim_{x\to c} f(x) = \lim_{x\to c} x =$ ______ , where c is any number.

Evaluate each of the following limits.

7. $\lim_{x\to 3} x =$ ______ 8. $\lim_{x\to 3} 6 =$ ______ 9. $\lim_{x\to 3}(x+6) =$ ______

10. $\lim_{x\to -5} 7 =$ ______ 11. $\lim_{x\to -5} x =$ ______ 12. $\lim_{x\to -5}(7-x) =$ ______

13. $\lim_{x\to 3}(6x) =$ ______ 14. $\lim_{x\to -5}(7x) =$ ______ 15. $\lim_{x\to 3}\left(\frac{x+6}{6x}\right) =$ ______

16. $\lim_{x\to -5}\left[(7x)\right]^2 =$ ______ 17. $\left[\lim_{x\to -5}(7x)\right]^2 =$ ______

18. $\lim_{x\to 3}\sqrt{x+6} =$ ______ 19. $\sqrt{\lim_{x\to 3} x+6} =$ ______

20. $\lim_{x\to 2}\left[3x^4 - x^3 + 2x^2 + x - 1\right] =$ ______

Use your results from above to complete the following:

21. $\lim_{x\to c}\left[f(x)+g(x)\right] =$ ______________________________

In words, the limit of the sum of two functions equals ____.

22. $\lim_{x\to c}\left[f(x)-g(x)\right] =$ ______________________________

In words, the limit of the difference of two functions equals ____.

23. $\lim_{x\to c}\left[f(x)\bullet g(x)\right] =$ ______________________________

In words, the limit of the product of two functions equals ____.

24. If $\lim_{x\to c} f(x)$ is known and if $n \geq 2$ is a positive integer, then

$\lim_{x\to c}\left[f(x)\right]^n =$ ______________________ and $\lim_{x\to c}\sqrt[n]{f(x)} =$ __________________________.

In words, the limit of the power a function equals ____.`

25. $\lim_{x\to c}\left[\frac{f(x)}{g(x)}\right] =$ ______________________

In words, the limit of the quotient of two functions equals ____.

26. If P is a polynomial function then, $\lim_{x\to c} P(x) =$ ______________________ .

Finding Limits Analytically: <u>Techniques:</u>

1. **<u>Substitution Theorem:</u>** For a polynomial or a rational function, $\lim_{x\to c} f(x) = f(c)$, if the denominator of the rational function is not zero. If substitution yields $\frac{0}{0}$, the following may be needed in order to find the limit.

<u>Examples Direct Substitution</u>

A. $\lim_{x\to 5}(x-7)$ B. $\lim_{x\to 5}(x^2-3x+2)$ C. $\lim_{x\to 5}\left(e^x+\pi x\right)$ D. $\lim_{x\to 3}\frac{x^4+3x^3-13x^2-27x+36}{x^2+3x-4}$

E. $\lim_{x\to 0}(x\cdot\cos(2x))$ F. $\lim_{x\to 2}(5x^2-3x+1)$

G. $\lim_{x\to 3}\frac{1}{x-3}$ H. $\lim_{\Delta x\to 0}\frac{(3+\Delta x)^2-3^2}{\Delta x}$

Use algebra to simplify

numerator.

2. **<u>Factor Numerator and/or Denominator:</u>**

<u>Examples Factoring and Cancellation</u>

A. $\lim_{x\to 3}\frac{x^2-x-6}{x-3}$ B. $\lim_{x\to -2}\frac{x+2}{x^2-4}$

C. $\lim_{x\to 3}\frac{x^3-27}{x-3}$

D. $\lim_{x\to 1}\frac{x^2-1}{1-x^2}$

3. **<u>Rationalize the Numerator or Denominator using Conjugate</u>**

<u>Examples Multiply by the Conjugate</u>

A. $\lim_{x \to 0} \frac{2-\sqrt{4+x}}{x}$ B. $\lim_{x \to 49} \frac{x-49}{\sqrt{x}-7}$ C. $\lim_{x \to 3} \frac{3-x}{\sqrt{1+x}-2}$ D. $\lim_{x \to 0} \frac{\sqrt{x+3}-\sqrt{3}}{x}$

4. <u>Complex Fractions</u>

<u>Examples Simplify Complex Fractions (multiply by 1 as a fraction equal to LCD/LCD)</u>

A. $\lim_{x \to 4} \frac{\left(\frac{1}{x}-\frac{1}{4}\right)}{(x-4)}$ B. $\lim_{x \to 6} \frac{\left(\frac{1}{x-5}-1\right)}{(x-6)}$ C. $\lim_{x \to 2} \frac{\frac{1}{x}-\frac{1}{2}}{x-2}$

D. $\lim_{x \to 2} \frac{1-\frac{2}{x}}{x^2-4}$ E. $\lim_{a \to 0} \frac{\left(\frac{1}{a+x}-\frac{1}{x}\right)}{(a)}$

For Exercises 3-8, find the limits if $\lim_{x \to c} f(x) = 2$ **and** $\lim_{x \to c} g(x) = -3$

3. $\lim_{x \to c} \sqrt{2f(x) - 4g(x)}$	4. $\lim_{x \to c} \left[f(x) + 1 \right]^3$	5. $\lim_{x \to c} \frac{2f(x) + 3g(x)}{g(x) - f(x)}$
6. $\lim_{x \to c} \left[f(x) \bullet 2g(x) \right]$	7. $\lim_{x \to c} \left[f(x) \bullet \left(g(x) + 3 \right) \right]$	8. $\lim_{x \to c} \frac{f(x)^2}{1 - g(x)}$

AB Calculus LIMITS PRACTICE – PRACTICE - PRACTICE No calculator!

Find the limits if they exist. Show work in the space provided. Write your final answer in the right column…HERE!

1	$\lim_{x\to 3}\sqrt{9-x^2}$	
2 *	$\lim_{x\to -2}\left(\frac{x^3+8}{x+2}\right)$	
3	$\lim_{x\to 0}\left(\frac{\frac{1}{x+3}-\frac{1}{3}}{x}\right)$	
4 *	$\lim_{x\to 27}\left(\frac{\sqrt[3]{x}-3}{x-27}\right)$	
5	$\lim_{x\to 0}\left(\frac{1}{x^3}\right)$	
6	$\lim_{x\to 100}\frac{\sqrt{x}-10}{x-100}$	
7 *	$\lim_{x\to \infty}\left(\frac{4x+3}{\sqrt{16x^2-2}}\right)$ $\lim_{x\to -\infty}\left(\frac{4x+3}{\sqrt{16x^2-2}}\right)$	
8	$\lim_{x\to 3}\left(\frac{2x^2-3x-9}{x-3}\right)$	
9	$\lim_{x\to 3}\left(\frac{2x^2-3x-8}{x-3}\right)$	
10	$\lim_{x\to 3}\left(\frac{2x^2-3x-9}{x-5}\right)$	
11	$\lim_{x\to 1}\left(\frac{1-\sqrt{4x^2-3}}{x-1}\right)$	
12	$\lim_{x\to -5}\left(\frac{\lvert x+1\rvert}{x+1}\right)$ $\lim_{x\to -1}\left(\frac{\lvert x+1\rvert}{x+1}\right)$	

#14-16 Find all vertical and horizontal asymptotes. Vertical Horizontal/Oblique

14. $f\left(x\right)=\dfrac{5x^2-3x+2}{36-21x^2+3x^4}$ ______________________________

15. $f\left(x\right)=\dfrac{2x+1}{\sqrt{9x^2-1}}$ ______________________________

16. $f\left(x\right)=\dfrac{x^3+5x+1}{x^2-5x+6}$ ______________________________

#17-28: Use this information to find limits.

$$\lim_{x\to 3} f\left(x\right)=3,\ \lim_{x\to 3} g\left(x\right)=4,\ \lim_{x\to 3} h\left(x\right)=0,\ \lim_{x\to 3^+} j\left(x\right)=5,\ \lim_{x\to 3^-} j\left(x\right)=0,\ \lim_{x\to 3^+} k\left(x\right)=0,\ \lim_{x\to 3^-} k\left(x\right)=5$$

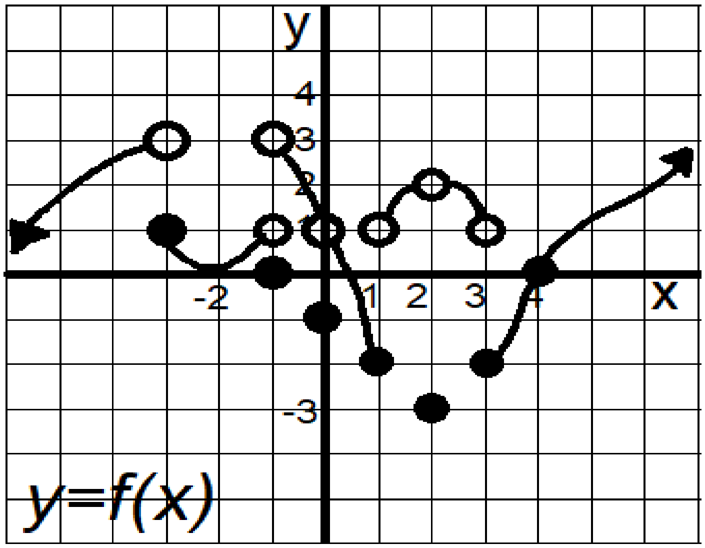
17. $\lim\limits_{x\to 3}\left[f\left(x\right)+g\left(x\right)\right]$	18. $\lim\limits_{x\to 3}\left[f\left(x\right)\bullet h\left(x\right)\right]$	19. $\lim\limits_{x\to 3}\left[\dfrac{h\left(x\right)}{f\left(x\right)}\right]$	20. $\lim\limits_{x\to 3}\left[\dfrac{g\left(x\right)}{h\left(x\right)}\right]$
21. $\lim\limits_{x\to 3}\left[j\left(x\right)\right]$	22. $\lim\limits_{x\to 3}\left[k\left(x\right)\right]$	23. $\lim\limits_{x\to 3}\left[j\left(x\right)+k\left(x\right)\right]$	24. $\lim\limits_{x\to 3}\left[j\left(x\right)\bullet k\left(x\right)\right]$
25. $\lim\limits_{x\to 3}\left[j\left(x\right)+h\left(x\right)\right]$	26. $\lim\limits_{x\to 3}\left[j\left(x\right)\bullet h\left(x\right)\right]$	27. $\lim\limits_{x\to 3}\left[f\left(g\left(x\right)-1\right)\right]$	28. $\lim\limits_{x\to 9}\left[j\left(\sqrt{x}\right)\right]$

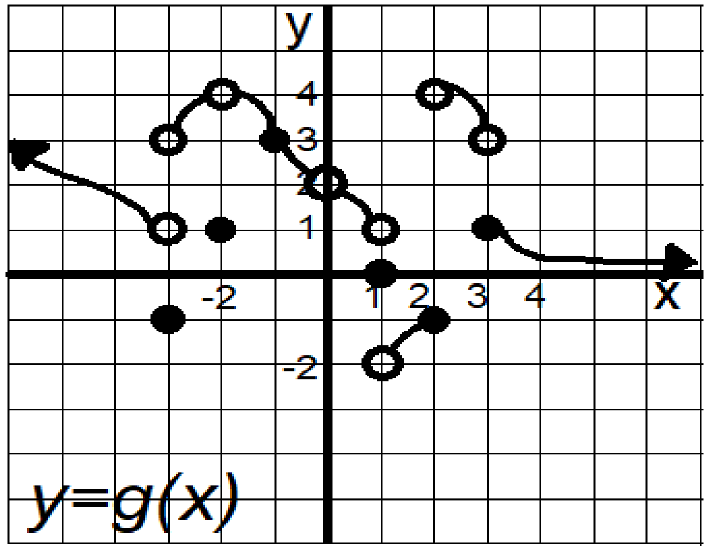
29) Draw pictures and write sentences to show two different ways the limit of a function will not exist.

More Work with Limits

Note that the limit properties (limit of a sum, difference, product, quotient of two functions) only apply if **the limits of both of the functions exist** at the specified point.

Practice: Given the functions *f(x)* and *g(x)* below, evaluate the limits or explain why they don't exist. NOTE: In the event you think the limit does not exist, be sure to check to see if the left-hand-limit equals the right-hand-limit to confirm your conclusion.





1. $\lim\limits_{x\to 0-}\left[f(x)+g(x)\right]$	2. $\lim\limits_{x\to 0+}\left[f(x)+g(x)\right]$	3. $\lim\limits_{x\to 0}\left[f(x)+g(x)\right]$	4. $\lim\limits_{x\to 0}\left[f(x)\bullet g(x)\right]$
5. $\lim\limits_{x\to 3}\left[f(x)+g(x)\right]$	6. $\lim\limits_{x\to 3}\left[f(x)\bullet g(x)\right]$	7. $\lim\limits_{x\to 3} f\left(g(x)\right)$	8. $\lim\limits_{x\to 0} f\left(g(x)\right)$
9. $\lim\limits_{x\to -3}\left[f(x)\bullet g(x)\right]$	8. $\lim\limits_{x\to 2} g\left(f(x)\right)$	11. $\lim\limits_{x\to -1} f\left(g(x)\right)$	12. $\lim\limits_{x\to 4} g\left(f(x)\right)$

Compute Limits Analytically using algebraic strategies, showing appropriate work and notation.

Strategies:
1) direct substitution
2) factor
3) rationalize using conjugate factor
4) simplify complex fractions.

<u>Examples</u>: Evaluate each of the following.

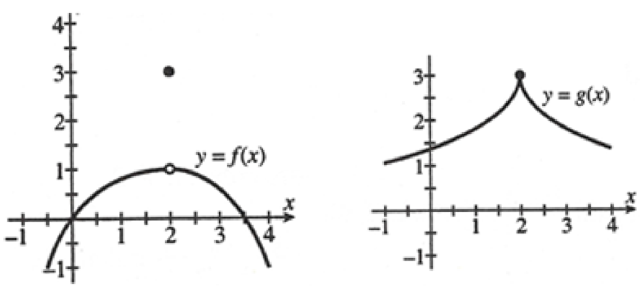
13. $\lim\limits_{x\to 6}\left(4x^2-10x+3\right)$

14. $\lim\limits_{x\to 1}\left(\dfrac{x^2-6x+5}{x-1}\right)$

15. $\lim\limits_{x\to 5}\sqrt[3]{\dfrac{4x+44}{6x-29}}$

16. $\lim\limits_{x\to 3} f(x)$ *where* $f(x)=\begin{cases} x^2 & x\geq 3 \\ 6x-4 & x<3 \end{cases}$

17. *given the graphs* $\lim\limits_{x\to 2}\left[f(x)+g(x)\right]$



18. $\lim\limits_{x\to 0}\left(\dfrac{\sqrt{2+x}-\sqrt{2}}{x}\right)$

19. $\lim\limits_{x\to 1}\left(\dfrac{\frac{1}{x}-x}{\frac{1}{x}-1}\right)$

20. $\lim\limits_{x\to 9}\dfrac{\sqrt{x}-3}{x-9}$

21. $\lim\limits_{x\to 5}\dfrac{5-x}{\left(\frac{1}{5}-\frac{1}{x}\right)}$

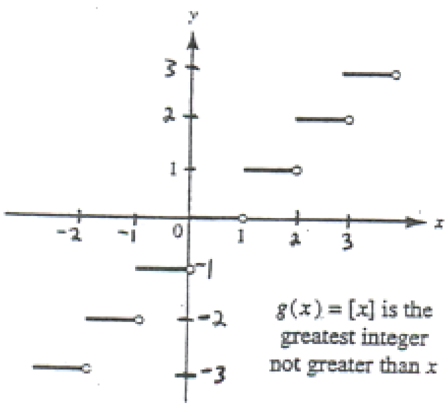
1.8 Limits--Student Notes   HH6ed

Part 1: Definition of Limits

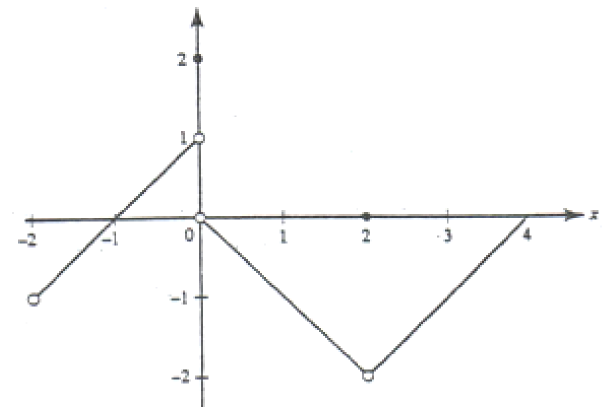
The number L is the *limit of the function f(x)* as x approaches c if, as the values of x get arbitrarily close (but not equal) to c, the values of $f(x)$ approach (or equal) L. We write $\lim_{x \to c} f(x) = L$.

In order for $\lim_{x \to c} f(x)$ to exist, the values of f must tend to the same number L as x approaches c from either the left or the right. We write $\lim_{x \to c-} f(x)$ for the *left-hand limit* of f at c (as x approaches c through values *less* than c), and $\lim_{x \to c+} f(x)$ for the *right-hand limit* of f at c (as x approaches c through values *greater* than c),

Example 1:



Example 2:



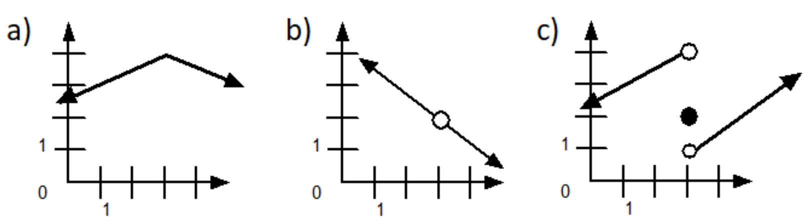
Find the following for Example 1:

1. $\lim_{x \to 0.6} x$
2. $\lim_{x \to -2.9} x$
3. $\lim_{x \to 2^-} x$
4. $\lim_{x \to 2^+} x$
5. $\lim_{x \to 2} x$

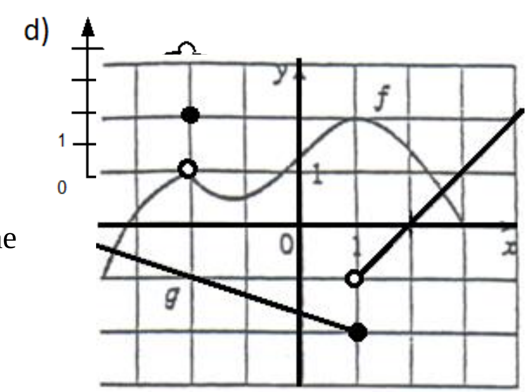
Find the following for Example 2:

6. $\lim_{x \to 3} f(x)$
7. $\lim_{x \to 2} f(x)$
8. $\lim_{x \to 0^-} f(x)$
9. $\lim_{x \to 0^+} f(x)$
10. $\lim_{x \to 0} f(x)$

Example 3: Given the graphs of each function, state whether or not $\lim_{x \to 3} f(x)$ exists and, if it does, give its value.



Use the graphs of f and g in the figure at the right to evaluate the limits, if they exist.



11. $\lim_{x\to -2}\left[f(x)+5g(x)\right]$

12. $\lim_{x\to 1}\left[f(x)\cdot g(x)\right]$

13. $\lim_{x\to 2}\left[\dfrac{f(x)}{g(x)}\right]$

14. $\lim_{x\to 3}\left[g^2(x)\right]$

NOTE: <u>Part 2: has been moved to “Applying Limit Properties”</u>

<u>Part 3: Limits at Infinity</u>

To find $\lim_{x\to\infty}\dfrac{f(x)}{g(x)}$, where *f*(*x*) and *g*(*x*) are polynomials in *x*, we can divide both numerator and denominator by the highest power of *c* that occurs and use the fact that $\lim_{x\to\infty}\dfrac{1}{x}=0$.

Examples:

31. $\lim_{x\to\infty}\dfrac{3-x}{4+x+x^2}$

32. $\lim_{x\to\infty}\dfrac{4x^4+5x+1}{37x^3-9}$

33. $\lim_{x\to\infty}\dfrac{x^3-4x^2+7}{3-6x-2x^3}$

<u>Rational Function Theorem:</u> Given $\lim_{x\to\infty}\dfrac{f(x)}{g(x)}$ (also applies when $x\to-\infty$)

i. when the degree of *f*(*x*) < *g*(*x*)____

ii. when the degree of *f*(*x*) = *g*(*x*)____

iii. when the degree of *f*(*x*) > *g*(*x*)____

<u>*Practice</u>:

34. $\lim_{x\to\infty} \frac{2x+1}{x-2}$

35. $\lim_{x\to-\infty} \frac{x^2+2x-3}{2x^3}$

36. $\lim_{x\to-\infty} \frac{x}{x^2+3}$

37. $\lim_{x\to\infty} \frac{13-2x}{3x+2}$

38. $\lim_{x\to-\infty} \frac{x^3-5}{1-x}$

39. $\lim_{x\to\infty} \frac{x^3-4x^4+12}{2x^4-1}$

40. Complete the table

Function	Limit as $x \to 0$	Limit as $x \to +\infty$	Limit as $x \to -\infty$
1. $f(x)=\lvert x\rvert$			
2. $f(x)=\frac{\lvert x\rvert}{x}$			
3. $f(x)=7$			
4. $f(x)=\cos x$			
5. $f(x)=\frac{1}{x}$			
6. $f(x)=\frac{x+4}{x-2}$			
7. $f(x)=\frac{x-2}{x^2-4}$			
8. $f(x)=\frac{x^2-4}{x-2}$			

41. Sketch the graph of a function f that satisfies all of the following conditions:

a. Its domain is the interval [0, 4]

b. $f(0) = f(1) = f(2) = f(3) = f(4) = 1$

c. $\lim_{x \to 1} f(x) = 2$

d. $\lim_{x \to 2} f(x) = 1$

e. $\lim_{x \to 3} f(x) = 2$

42. Let $f(x) = \dfrac{2x^2 + x + k}{x^2 - 2x - 3}$ and $g(x) = \dfrac{5x^2 + 12x + k}{x^2 - 2x - 3}$.

Find the number k so that each limit exists?

Evaluate each limit and justify your answer.

a. $\lim_{x \to 3} f(x)$

b. $\lim_{x \to -1} f(x)$

c. $\lim_{x \to \infty} f(x)$

d. $\lim_{x \to 3} g(x)$

e. $\lim_{x \to -1} g(x)$

f. $\lim_{x \to \infty} g(x)$

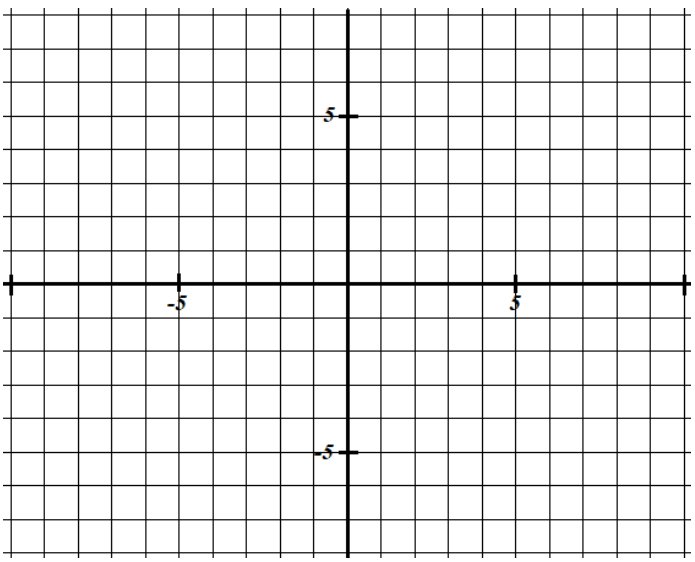
43. Sketch the graph of a function satisfying the following conditions.

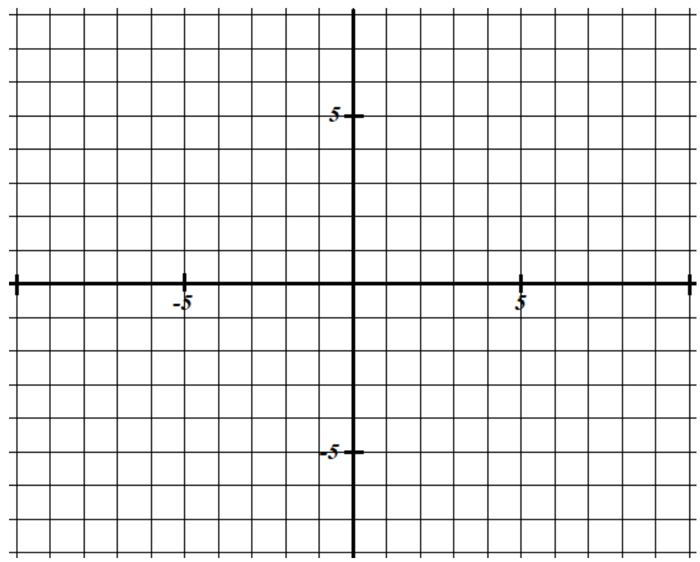
Graph A

1) $f(x)$ is an EVEN function (symmetry to the y-axis).
2) $f(x)$ has a single root at $x=2$, double root at $x=4$.
3) $f(1)=6$
4) $\lim_{x\to 1} f(x) = -3$
5) $\lim_{x\to 0} f(x) = -7$
6) $\lim_{x\to 6^-} f(x) = \infty$
7) $\lim_{x\to 6^+} f(x) = -\infty$
8) $\lim_{x\to \infty} f(x) = 0$

Graph B

1) $f(x)$ is an ODD function (symmetry to the origin).
2) $f(x)$ has a single root at $x=0$, triple root at $x=4$.
3) $f(2)=-3$
4) $\lim_{x\to 2} f(x) = 4$
5) $\lim_{x\to 6} f(x) = -\infty$
6) $\lim_{x\to \infty} f(x) = \infty$
7) $f(x)$ has an oblique asymptote





Evaluating Limits Analytically

Find each of the following limits analytically.

4. $\lim_{x\to 3} \frac{x^3-27}{x-3}$

5. $\lim_{x\to 5} \frac{x^2-5x}{x^2-25}$

6. $\lim_{x\to -1} \frac{x^{89}+2x^{56}-6x^{18}-8x-7}{x^{59}+3x^{46}-17x^{15}+19}$

7. $\lim_{x\to 4} e$

8. $\lim_{x\to 1} \frac{(x-1)^4}{x^4-1}$

9. $\lim_{x\to 0} \frac{2-\sqrt{4+x}}{x}$

10. $\lim_{x\to 0} \frac{3x^3-2x^2+x-7}{8x^3+x^2-9x+7}$

11. $\lim_{x\to 2} \frac{1-\frac{2}{x}}{x^2-4}$

For Exercises 14 - 19, find the limits if $\lim_{x\to c} f(x)=2$ and $\lim_{x\to c} g(x)=-3$

14. $\lim_{x\to c} \sqrt{2f(x)-4g(x)}$

15. $\lim_{x\to c} [f(x)+1]^3$

16. $\lim_{x\to c} \frac{2f(x)+3g(x)}{g(x)-f(x)}$

17. $\lim_{x\to c} [f(x) \cdot 2g(x)]$

18. $\lim_{x\to c} [f(x) \cdot (g(x)+3)]$

19. $\lim_{x\to c} \frac{f(x)^2}{1-g(x)}$

Concept of Derivative: Vocabulary, Slope of Secant, Slope of Tangent

I. **Complete the table by matching** each of the following descriptions with an appropriate graph and table of values.

Description	Table	Graph
A		
B		
C		
D		

A) The weight of your jumbo box of Fruity Flakes decreases by an equal amount every week.
B) The machinery depreciated rapidly at first, but its value declined more slowly as time went on.
C) In free fall, your distance from the ground decreases at an increasing rate.
D) For a while it looked like the decline in profits was slowly down, but then began declining ever more rapidly.

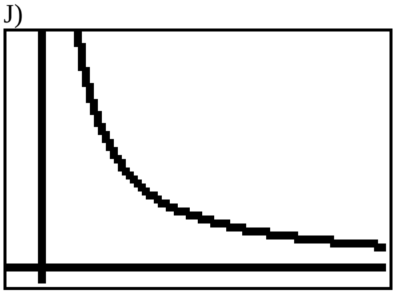
E) x	0	1	2	3	4	5
y	400	384	336	256	144	0

F) x	0	1	2	3	4	5
y	400	320	240	160	80	0

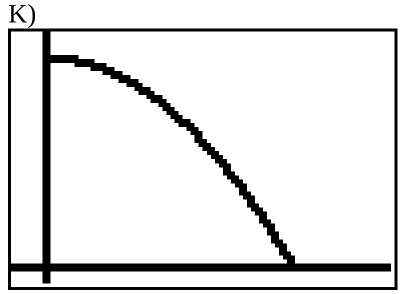
G) x	0	1	2	3	4	5
y	400	184	98	63	49	43

H) x	0	1	2	3	4	5
y	412	265	226	224	185	38

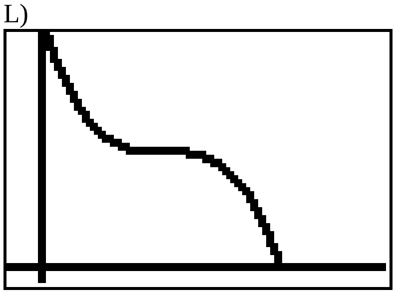
J)



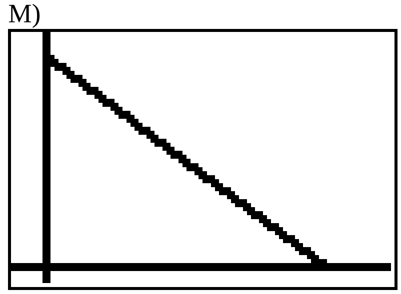
K)



L)



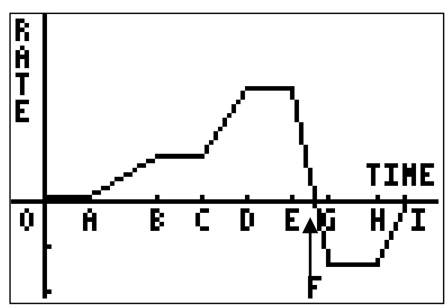
M)



II. Interpretation: The graph represents the **RATE** at which the volume of water in a reservoir is changing for time $t > 0$.

What is happening to the volume of water in the reservoir if the rate is negative?____

For each of the following statements, give the largest interval on which:



A) The volume of the water is increasing.____________________

B) The volume of the water is constant.______________________

C) The volume of the water is increasing the fastest.___________

D) The volume of the water is decreasing.___________________

On what intervals is the water level in the reservoir not changing____________________

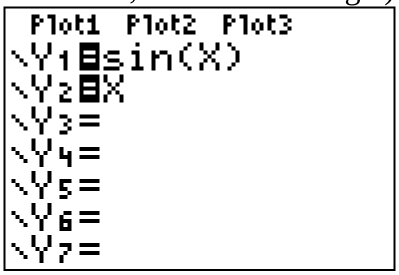
Increasing at a constant rate______________
Increasing at a increasing rate_____________
Increasing at a decreasing rate____________

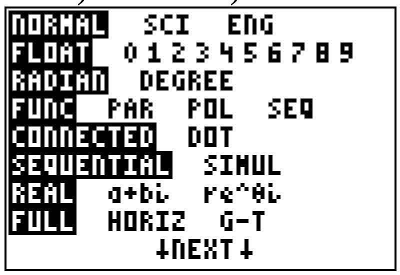
Decreasing at an constant rate____________
Decreasing at an increasing rate___________
Decreasing at an decreasing rate__________

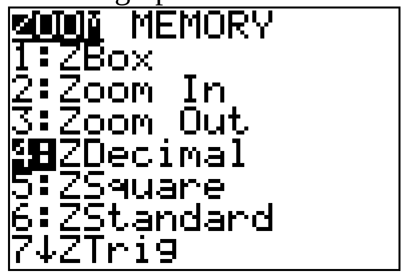
TWO TRIG LIMITS EXPLORATION: Let's examine the following limit using technology to help us evaluate.

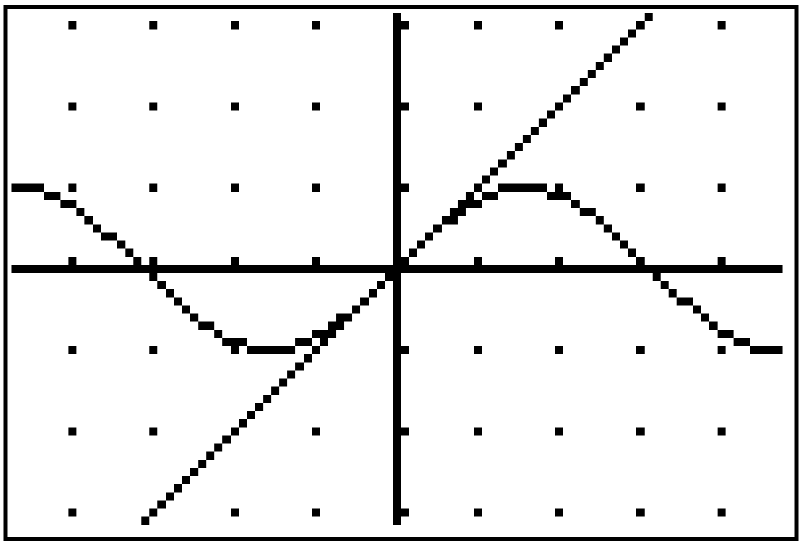
$$\#1)\quad \lim_{x\to 0}\left(\frac{\sin x}{x}\right)=?$$

Let's look at the graphs of the numerator and denominator separately, on the same graph. On your graphing calculator, set the following 1) Y= 2) MODE 3) ZOOM to obtain the graph.







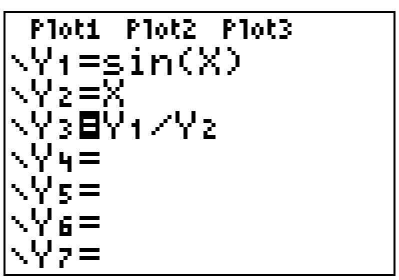


Notice that very close to zero the graph of y=sin(x) and y=x look very much the same.

So what value would you assign to

$$\lim_{x\to 0}\left(\frac{\sin x}{x}\right)=?$$

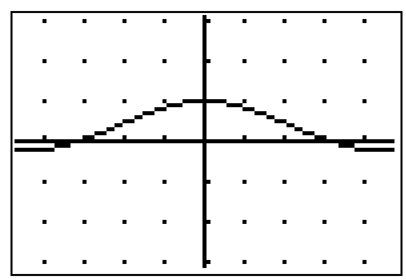
In Y= turn off Y1 and Y2. Enter Y3 as shown. GRAPH and evaluate



$$\lim_{x\to 0}\left(\frac{\sin x}{x}\right)=?$$

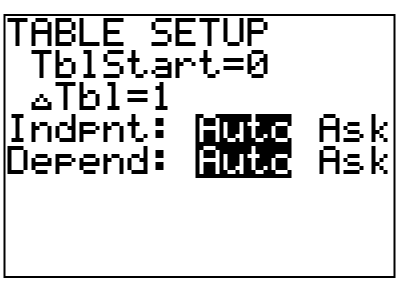
TRACE to x = 0. Move the left and right arrow keys to the left and right of x=0.

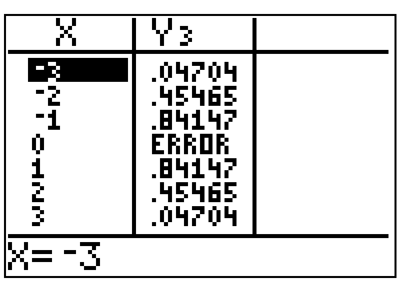
What appears to be happening at this x-value?



Press 2nd ZOOM to <u>turn off the axes</u> and GRAPH again. Now what do you see, or don't you see?

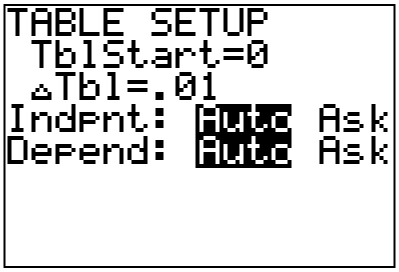
Let's examine the TABLE before we conclude. Press 2nd WINDOW to get to the TABLE SETUP menu. Set it as shown. Press 2nd GRAPH to see the table and use the up arrow to scroll to x= –3.





Examine the y-values in the table as x gets closer to zero from the left and as x gets closer to zero from the right.
What appears to be happening?

Return to the Table SetUp. Reset the TblStart =0. Now let’s zoom in on the table by changing the Table Step shown as ΔTbl from 1 to 0.01.



Press 2nd GRAPH to view the table again.

Examine the y-values in the table as x gets closer to zero from the left and as x gets closer to zero from the right.
What appears to be happening?

X	Y3
-.03	.99985
-.02	.99993
-.01	.99998
0	ERROR
.01	.99998
.02	.99993
.03	.99985

In conclusion, we now know graphically that

$$\boxed{\lim_{x\to 0}\left(\frac{\sin x}{x}\right) = \underline{\qquad\qquad}}$$

Now let’s examine the limits of variations of #1. Use your calculator to determine the values of these limits. All answers must be exact, no decimals. So use fraction form!

$a)\ \lim_{x\to 0}\left(\frac{\sin(8x)}{8x}\right) = ?$ $\qquad b)\ \lim_{x\to 0}\left(\frac{\sin(2x)}{2x}\right) = ?$ $\qquad c)\ \lim_{x\to 0}\left(\frac{\sin(x)}{6x}\right) = ?$

$d)\ \lim_{x\to 0}\left(\frac{\sin(3x)}{x}\right) = ?$ $\qquad e)\ \lim_{x\to 0}\left(\frac{\sin(9x)}{4x}\right) = ?$ $\qquad f)\ \lim_{x\to 0}\left(\frac{\sin(5x)}{7x}\right) = ?$

PRACTICE: Evaluate the following limits. <u>DO NOT USE your calculator.</u>

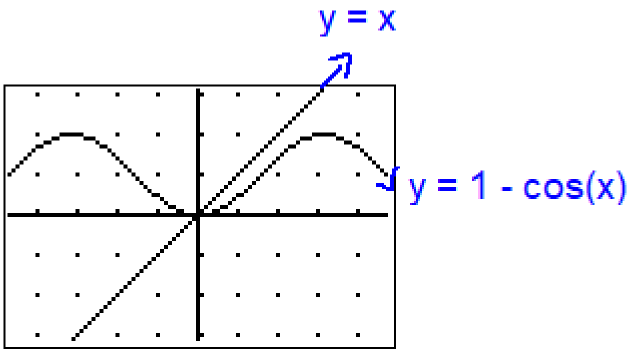
$G)\ \lim_{x\to 0}\left(\frac{\sin(3x)}{5x}\right) = ?$ $\qquad H)\ \lim_{x\to 0}\left(\frac{\sin(7x)}{2x}\right) = ?$ $\qquad I)\ \lim_{x\to 0}\left(\frac{\sin(3x)}{6x}\right) = ?$

$J)\ \lim_{x\to 0}\left(\frac{9\sin(x)}{2x}\right) = ?$ $\qquad K)\ \lim_{x\to 0}\left(\frac{\sin(12x)}{4x}\right) = ?$ $\qquad L)\ \lim_{x\to 0}\left(\frac{\sin(15x)}{10x}\right) = ?$

Let's examine the following limit using technology to help us evaluate.

$$\#2)\quad \lim_{x\to 0}\left(\frac{1-\cos x}{x}\right) = \underline{\qquad\qquad\qquad}?$$

After graphing Y1 = 1 – cos(x) and Y2 = x, what are your thoughts about this limit?



Let's take a closer look using the table.

We'll look at the table near x=0 with a two different ΔTbl zooming in toward zero.

1) ΔTbl=0.01 2) ΔTbl = 0.0001

For ΔTbl = 0.01, examine the LHL and the RHL.

$$\lim_{x\to 0-}\frac{1-\cos(x)}{x} \Rightarrow \left.\frac{1-\cos(x)}{x}\right|_{x=-0.01} = \frac{+0.\qquad\qquad\qquad}{-0.} = \underline{\qquad\qquad\qquad}\ldots$$

$$\lim_{x\to 0+}\frac{1-\cos(x)}{x} \Rightarrow \left.\frac{1-\cos(x)}{x}\right|_{x=+0.01} = \frac{+0.\qquad\qquad\qquad}{+0.} = \underline{\qquad\qquad\qquad}\ldots$$

For ΔTbl = 0.0001, examine the LHL and the RHL.

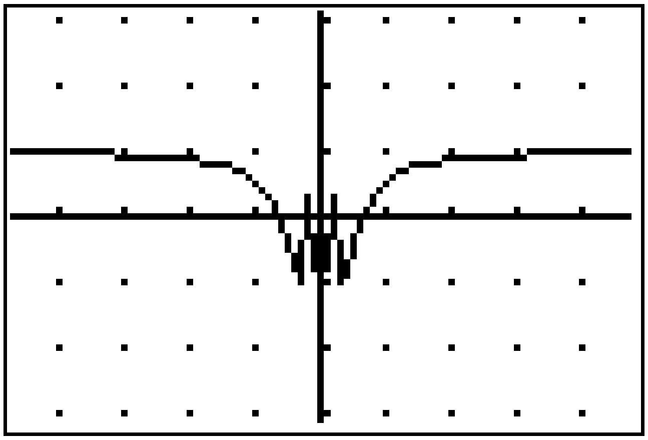
$$\lim_{x\to 0-}\frac{1-\cos(x)}{x} \Rightarrow \left.\frac{1-\cos(x)}{x}\right|_{x=-0.0001} = \frac{+0.\qquad\qquad\qquad}{-0.} = \underline{\qquad\qquad\qquad}\ldots$$

$$\lim_{x\to 0+}\frac{1-\cos(x)}{x} \Rightarrow \left.\frac{1-\cos(x)}{x}\right|_{x=+0.0001} = \frac{+0.\qquad\qquad\qquad}{-0.} = \underline{\qquad\qquad\qquad}\ldots$$

After closer examination of the left and right-hand limits as x approaches zero, the limit values appear to be getting closer and closer to what value?

In conclusion: $\boxed{\lim_{x\to 0}\left(\frac{1-\cos x}{x}\right) = \underline{\qquad\qquad}}$

Explore the behavior of the graph $f(x) = \cos\left(\frac{1}{x}\right)$ near $x = 0$ by using the following windows to zoom in near zero.



X: [-4.7, 4.7] Y: [-3.1,3.1]

X: [-2, 2] Y: [-1.2, 1.2]

X: [-1, 1] Y: [-1.2, 1.2]

X: [-0.5, 0.5] Y: [-1.2, 1.2]

X: [-0.25, 0.25] Y: [-1.2, 1.2]

X: [-0.10, 0.10] Y: [-1.2, 1.2]

Repeat to explore the behavior of the graph $f(x) = \sin\left(\frac{1}{x}\right)$ near $x = 0$.