

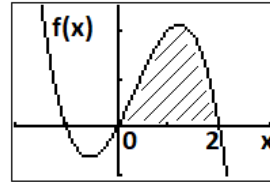
## Unit 6: Chapter 8 – Areas and Volumes & Density Functions

**Part 1: Areas** To find the area bounded by a function  $f(x)$  and the  $x$ -axis on an interval from  $x = a$  and

$x = b$  we use the integral:  $\int_a^b f(x) dx$ . This is consistent with  $\int_a^b (\text{upper} - \text{lower}) dx$  since

$$\int_a^b f(x) dx = \int_a^b [f(x) - 0] dx.$$

For example: 
$$\int_1^2 (2x + x^2 - x^3) dx = \int_1^2 (-x)(x+1)(x-2) dx$$



The examples below show the two options for the integral set up to find areas between curves.

<p style="text-align: center;">On the <math>x</math>-interval from <math>x = a</math> and <math>x = b</math>:</p> $\int_a^b (\text{upper curve} - \text{lower curve}) dx$ <div style="text-align: center;"> </div> $\int_{-1}^2 (f(x) - g(x)) dx = \int_{-1}^2 ((4 - x^2) - (2 - x)) dx$	<p style="text-align: center;">On the <math>y</math>-interval from <math>y = c</math> and <math>y = d</math>:</p> $\int_c^d (\text{right curve} - \text{left curve}) dy$ <div style="text-align: center;"> </div> $\int_{-1}^2 (f(y) - g(y)) dy = \int_{-1}^2 ((4 - y^2) - (2 - y)) dy$
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When the area is bounded by two curves, you may need to solve for the intersection points to find the limits of integration. For efficiency and accuracy, it is best to store the intersection values on the calculator. STO>A immediately then repeat and STO>B immediately.

**DAY 116 HW:** Draw and label an accurate graph, then either write an integral to find the area between the two functions from  $x = a$  and  $x = b$  or to find the area bounded by the two functions. For bounded regions, you must determine the limits of integration. Show work to evaluate the integral and find the area. Do your best to graph the functions without your graphing calculator when indicated “no calculator”.

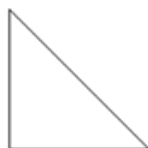
1. between $f(x) = x + 1$ $g(x) = \frac{1}{2}x$ $a = 0, b = 3$ No calculator	2. between $f(x) = 4 - x$ $g(x) = x^2 - x - 2$ $a = -1, b = 2$ No calculator	3. bounded by $f(x) = 4 - x^2$ $g(x) = \frac{1}{4}x^3 - x$ No calculator	4. bounded by $f(x) = e^x$ $g(x) = 2 - x^2$ Use calculator to find limits of integration.
5. bounded by $f(x) = \cos(x)$ $g(x) = \sin(x)$ Limits of integration may vary. No calculator	6. bounded by $f(x) = \sqrt{x}$ $g(x) = x^2$ Integrate with respect to $x$ . No calculator	7. bounded by $f(x) = \sqrt{x}$ $g(x) = x^2$ Integrate with respect to $y$ . No calculator	8. bounded by $f(x) = \cos(x)$ $g(x) = \sin(x)$ & $y$ -axis No calculator
9. bounded by $y = x - 1$ & $x = (y - 1)^2$ Integrate with respect to $y$ .	10. bounded by $f(x) = \sqrt{x}$ & $g(x) = 2\sqrt{5 - x}$ Integrate with respect to $y$ .		

**Part 2a: Volumes by Cross-Sections on a Base**

Record the area formulas for these shapes.



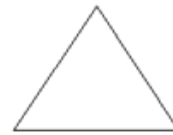
square



right isosceles triangle



semi-circle



equilateral triangle

To find the volume of a figure with a known cross sectional area, use the Riemann sum that becomes  $\int Area \, dx$  for a function.

1. The base of a solid is bound by  $y = \sqrt{x}$ ,  $y = 0$ , and  $x = 4$ .

a) Draw a picture of this base on each of the four the grids at the right.

b) Draw three segments in this base perpendicular to the x-axis. Repeat for the next three grids at the right.

c) What is the length of one representative segment in terms of x?

d) The segments you drew in part b)

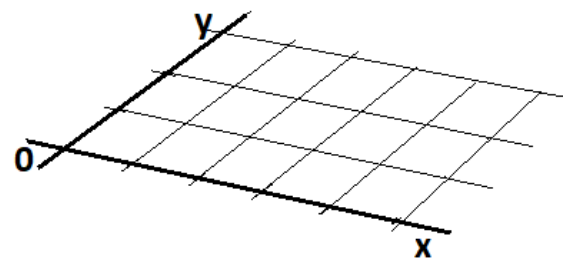
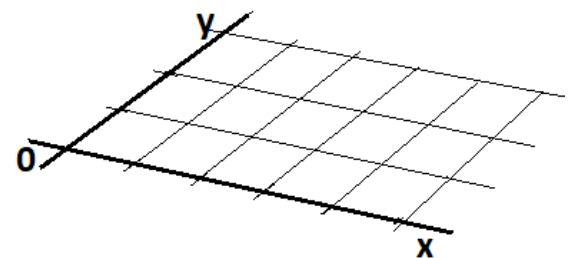
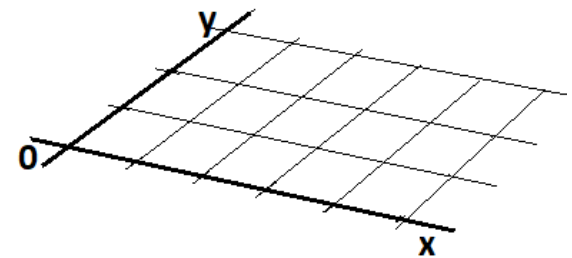
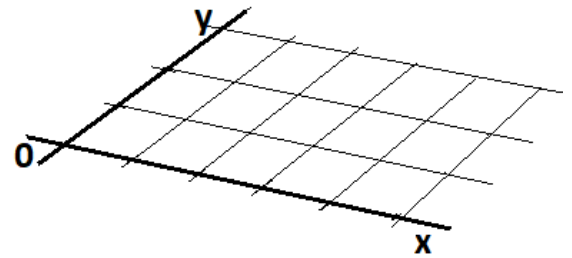
- o on the 1<sup>st</sup> grid are sides of a square. What is the area of the square in terms of x?
- o on the 2<sup>nd</sup> grid are legs of a right isosceles triangle. What is the area of the right isosceles triangle in terms of x?
- o on the 3<sup>rd</sup> grid are diameters of a semi-circle. What is the area of the semi-circle in terms of x?
- o on the 4<sup>th</sup> grid are sides of an equilateral triangle. What is the area of the equilateral triangle in terms of x?

e) Write an integral to find the volume formed by the solid with the given base area if the cross-section areas perpendicular to the x-axis are

Squares	Right Isosceles triangles
Semi-Circles	Equilateral triangles

f) What would change if the cross-sections were perpendicular to the y-axis?

g) Find the volume formed by the solid whose side length is bound by the base area and whose cross-section areas perpendicular to the y-axis are i) squares, ii) right isosceles triangles, iii) semi-circles, iv) equilateral triangles.



## Volumes of Known Cross-Sections

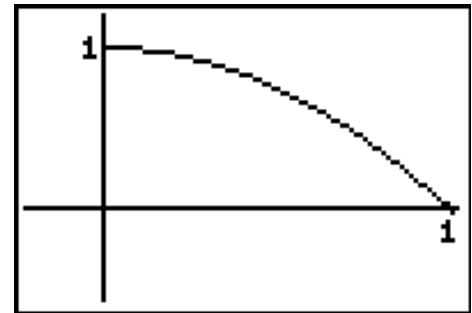
*Work Smarter not Harder Calculator Lab (TI-83)*

We know from geometry that we can find the volume of an object by multiplying its base area by its height. Consider a solid which has the property that cross-sections, perpendicular to a given line, have a known area.

If  $A(x)$  denotes the area of the cross section of the solid  $S$  for  $a \leq x \leq b$  and, if the function  $A(x)$  is continuous on  $[a, b]$ , then the volume  $V$  of  $S$  is  $V = \int_a^b A(x) dx$

One of the biggest difficulties in doing these problems is visualizing the solid. The program CROSSECT can help you visualize the volumes of solids whose cross-sections are squares or isosceles right triangles which are perpendicular to the x-axis. Consider the following example.

**Example 1:** Suppose the base of a solid is the region bounded by  $y = 1 - x^2$  and the x-axis on the interval  $[0, 1]$ . All cross-sections perpendicular to the x-axis are squares. Find the volume of the solid.



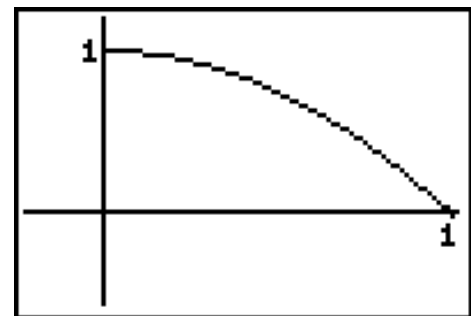
**Solution:** Begin by drawing the base of the region that forms the base of the solid, as shown at the right. In order to use the program CROSSECT to visualize the solid, perform the following steps. Run the program CROSSECT. Press [ENTER] Select 1:ENTER F(X) and enter  $1 - x^2$ . Press [ENTER]. Select 2: ENTER BOUNDS and enter LOWER = 0 and UPPER = 1. Select 3: SIDE SQUARE. Press [ENTER] repeatedly to see additional cross-sections until 5 cross-sections are displayed. Sketch the cross-sections on the diagram provided.

In order to find the volume of this solid, we must find an expression for the area of any cross-sectional square we draw. The formula for the area of a square with side,  $s$ , is  $s^2$ . So, we must define the side of any square we inscribe in our diagram. What is the length of the side of the square?

So,  $A(x) = s^2 = (1 - x^2)^2$  and the volume is  $V = \int_0^1 (1 - x^2)^2 dx$ . Use your calculator to evaluate this

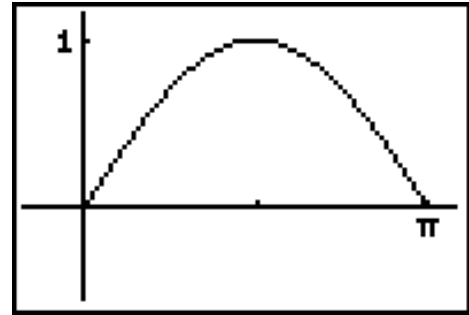
expression. \_\_\_\_\_ . You can check your answer by selecting 5: ANSWER in the program.

**Example 2:** Suppose the base of a solid is the region bounded by  $y = 1 - x^2$  and the x-axis on the interval  $[0, 1]$ . All cross-sections perpendicular to the x-axis are isosceles right triangles, where the right angle is at to the x-axis. Find the volume of the solid.



- a. Draw the diagram.
- b. Set up the volume integral.
  
- c. Evaluate the integral to find the answer.

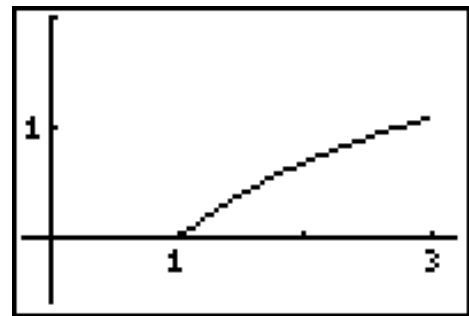
Example 3: Suppose the base of a solid is the region bounded by  $y = \sin x$  and the x-axis on the interval  $[0, \pi]$ . All cross-sections perpendicular to the x-axis are squares. Find the volume of the solid.



- a. Draw the diagram.
- b. Set up the volume integral.

- c. Evaluate the integral to find the answer.

Example 4: Suppose the base of a solid is the region bounded by  $y = \ln x$  and the x-axis on the interval  $[1, 3]$ . All cross-sections perpendicular to the x-axis are isosceles right triangles, where the right angle is at to the x-axis. Find the volume of the solid.



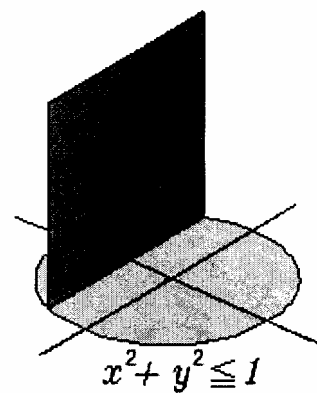
- a. Draw the diagram.
- b. Set up the volume integral.

- c. Evaluate the integral to find the answer.

Example 5: Suppose the base of a solid is the region bounded by  $x^2 + y^2 = 1$  and every cross-section perpendicular to the x-axis is a square. Find the volume of the solid. Be sure to show the integral, as well as, state the numerical answer.

- a. Draw the diagram.
- b. Set up the volume integral.

- c. Evaluate the integral to find the answer.

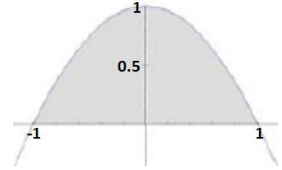


**DAY 117 HW #1-8 below. Show all work on your own paper.**

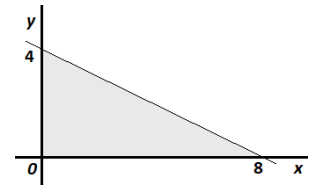
For each of the following, graph the region of the base, set up the integral to find the appropriate volume, then use your calculator to evaluate the integral. Answers are provided.

1. Find the volume of the solid with circular base of diameter 10 cm and whose cross-sections perpendicular to a diameter across the x-axis are equilateral triangles. Answer: 288.675

2. The base of a solid is the region bounded by the graph of  $y = 1 - x^2$  and the x-axis. For this solid, each cross section perpendicular to the x-axis is a rectangle with height three times the base. What is the volume of this solid? Answer: 3.2



3. The base of a solid is the region in the first quadrant bounded by the x-axis, the y-axis, and the line  $x + 2y = 8$ , as shown in the figure. If cross sections of the solid perpendicular to the x-axis are semicircles, what is the volume of the solid? Answer: 16.755



4. The base of a solid is bounded by  $y = \cos(x)$ , the x-axis,  $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$ . Cross sections perpendicular to the x-axis are squares. Find the volume. Calculate this one by hand. Answer:  $\frac{\pi}{2}$

5. The base of a solid is bounded by  $y = 2 - x$ , the x-axis, and the y-axis. Cross sections that are perpendicular to the x-axis are isosceles right triangles with the right angle on the x-axis. (Legs perpendicular to the x-axis). Find the volume. Answer:  $\frac{4}{3}$

6. The base of a solid is bounded by the semi-circle  $y = \sqrt{4 - x^2}$  and the x-axis. Cross sections that are perpendicular to the x-axis are squares. Find the volume. Answer:  $\frac{32}{3}$

7. The base of a solid is bounded by  $y = \sqrt{16 - x^2}$  and the x-axis. Cross sections that are perpendicular to the y-axis are equilateral triangles. Find the volume. Answer: 73.899

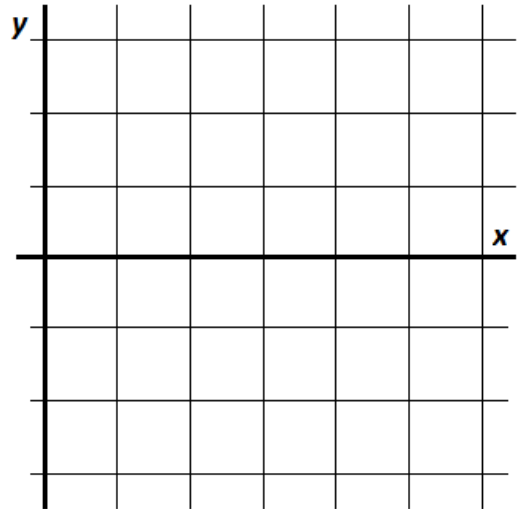
8. The base of a solid is bounded by  $y = 2 - \frac{1}{2}x$ , the x-axis, and the y-axis. Cross sections that are perpendicular to the y-axis are isosceles right triangles with the hypotenuse in the xy-plane. Find the volume. Answer:  $\frac{8}{3}$

**Part 2b: Volumes of Revolution (Disks)**

**Example 1:** The region between the curve  $y = \sqrt{x}$ ,  $0 \leq x \leq 4$  and the x-axis is revolved around the x-axis to generate a solid. Find its volume.

Follow these steps to visualize the volumes and process the algebraic solution:

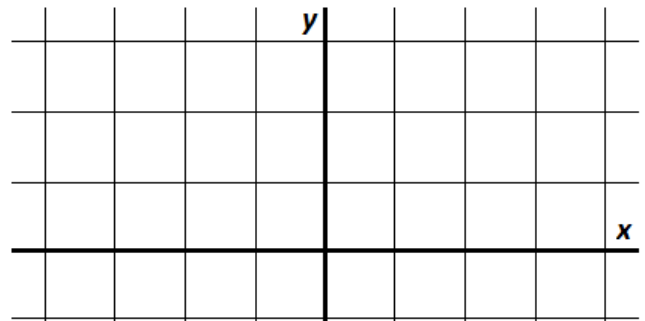
- Draw the bounded region described.
- Reflect the region in the x-axis.
- Draw three representative circular slices or disks perpendicular to the axis of rotation.
- Identify the radius of each disk in terms of the variable x.
- Identify the area of each disk in terms of the variable x.
- Disks perpendicular to a horizontal axis of rotation will have a dx-thickness.
- Identify the volume of each disk in terms of the variable x.
- Write an integral to sum the volume of all the disks.



Show work here:

**Example 2:** The region between the curve  $y = \sqrt{x}$ ,  $0 \leq x \leq 4$  and the y-axis is revolved around the y-axis to generate a solid. Find its volume.

Follow the steps outlined above. Since the region is revolved about the y-axis the disks are now perpendicular to the y-axis and they have a dy-thickness.



Show work here:

**Volume of Revolution by Disks**

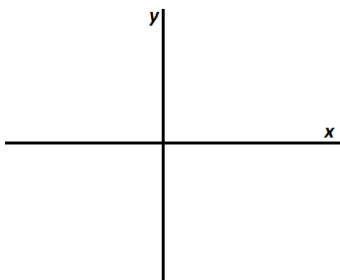
**Volume of a solid obtained by revolving a bounded region about an axis of rotation:**

$$V = \int_a^b \pi [R(x)]^2 dx \quad \text{or} \quad V = \int_c^d \pi [R(y)]^2 dy$$

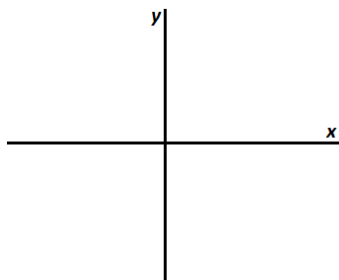
*for*  $a \leq x \leq b$                       *for*  $c \leq y \leq d$

*Problems:* Draw the picture and find the volume of the solid generated by revolving the region about the stated axis.

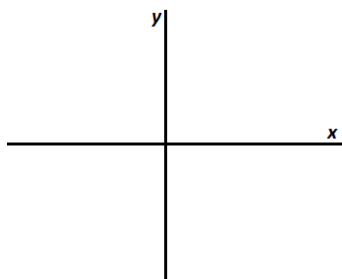
1.  $y = e^{-x}$ ,  $y = 0$ ,  $x = 1$ ,  $x = \ln(10)$   
 $x$ -axis



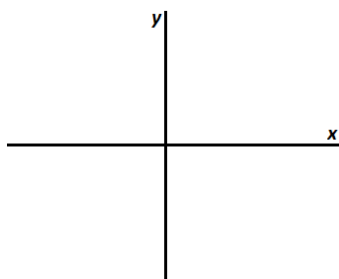
2.  $y = \sqrt{x}$ ,  $y = 0$ ,  $x = 4$   
axis:  $x = 4$



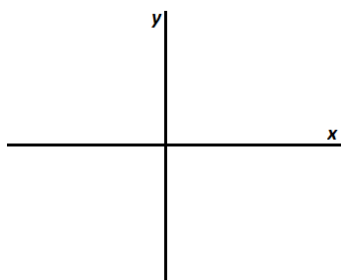
3.  $y = x^2$ ,  $y = 4$   
axis:  $y = 4$



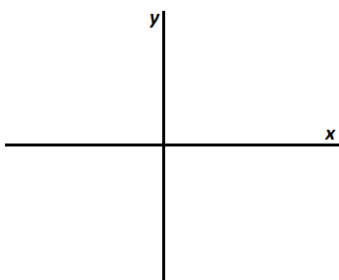
4.  $9x^2 + 25y^2 = 225$   
 $x$ -axis



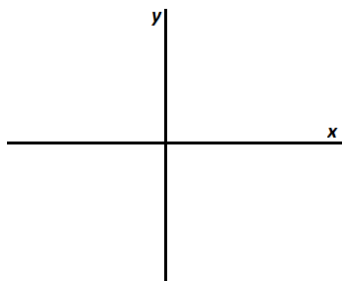
5.  $y = 4 - x^2$ ,  $y = 0$ ,  $x = -2$ ,  $x = 0$   
 $x$ -axis



6.  $y = \cos(x)$ ,  $y = 0$ ,  $x = 0$ ,  $x = \frac{\pi}{2}$   
 $x$ -axis



7. a circle with radius "r"  
y-axis



**DAY 118 HW Volumes of Revolution by Disks: #1-2 below and AP FRQ #3-4-5 next page.**

1. (Non-Calculator) Find the volume of the solid formed by revolving the shaded region in the X-AXIS.  
Show all work.

<p>a. <math>y = -x + 1</math>, ans: <math>\frac{\pi}{3}</math></p>	<p>b. <math>y = \sqrt{4 - x^2}</math>, ans: <math>\frac{16\pi}{3}</math></p>	<p>c. <math>y = \sqrt{x}</math>, ans: <math>\frac{15\pi}{2}</math></p>
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2. (Calculator) Find the volumes generated when the regions bounded by the given curves and lines are rotated about the Y-AXIS.

<p>a. <math>y = \frac{x}{2}</math>, <math>x = 0</math>, <math>y = 2</math> ans: <math>\frac{32\pi}{3}</math></p>	<p>b. <math>x = \sqrt{4 - y}</math>, <math>x = 0</math>, <math>y = 0</math> ans: <math>8\pi</math></p>	<p>c. <math>x = 1 - y^2</math>, <math>x = 0</math> ans: <math>\frac{16\pi}{15}</math></p>
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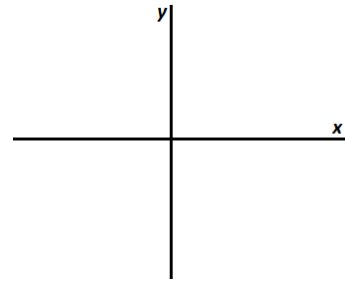
**DAY 118 HW Volumes of Revolution by Disks:** AP FRQs Calculator Active #3-4 NON-Calculator #5

3. Let  $R$  be the region bounded by the graph of  $y = \frac{1}{x} \ln(x)$ , the  $x$ -axis, and the line  $x = e$ .

(a) Draw the region described.

(b) Find the area of the region  $R$ .

(c) Find the volume of the solid formed by revolving the region  $R$  about the  $x$ -axis.



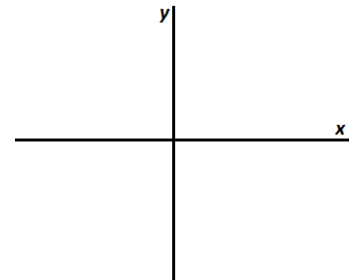
4. Let  $R$  be the region enclosed by the graph of  $y = \ln(x)$ , the line  $x = 3$ , and the  $x$ -axis.

(a) Graph the region described.

(b) Find the area of region  $R$ .

(c) Find the volume of the solid generated by revolving region  $R$  about the  $x$ -axis.

(d) Set up, but do not evaluate, an integral expression in terms of a single variable for the volume of the solid generated by revolving region  $R$  about the line  $x = 3$ .



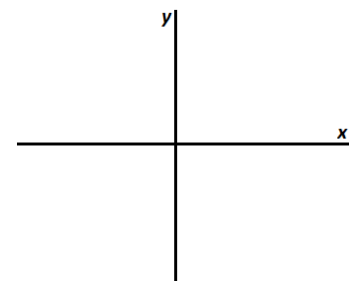
5. (Non-Calculator) Let  $R$  be the region in the first quadrant that is enclosed by the graph of  $y = \tan(x)$ , the  $x$ -axis, and the line  $x = \frac{\pi}{3}$ .

a. Graph the region  $R$ .

b. Find the area of  $R$ . Answer:  $\ln 2$

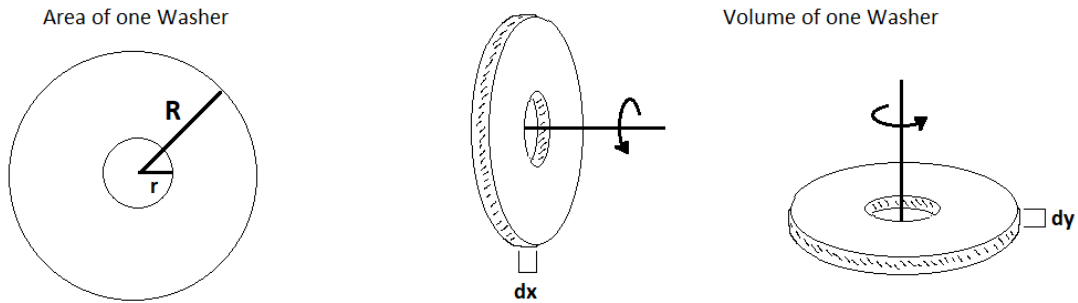
b. Find the volume of the solid formed by revolving the region  $R$  about the  $x$ -axis.

Answer:  $\pi \left( \sqrt{3} - \frac{\pi}{3} \right)$



**Part 2c: Volumes of Revolution (Washers)**

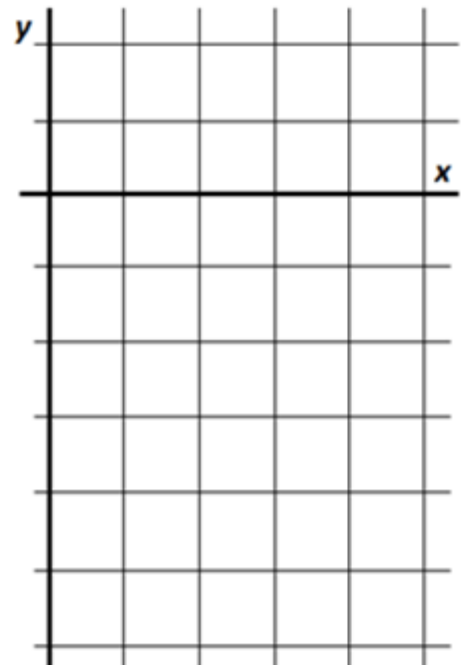
Sometimes slicing a solid of revolution results in disks with holes in the middle—washers. In order to find these volumes, it is necessary to subtract the volume of the hole.



**Example 1:** The region between the curve  $y = \sqrt{x}$ ,  $0 \leq x \leq 4$  and the  $x$ -axis is revolved around the horizontal line  $y = -2$  to generate a solid. Find its volume.

Follow these steps to visualize the volumes and process the algebraic solution:

- Draw the bounded region described. Draw the axis of rotation.
- Reflect the region in the axis of rotation.
- Draw three representative circular washer slices perpendicular to the axis of rotation.
- Identify the **Outer-R** and **inner-r** radius of each washer in terms of the variable  $x$ .
- Identify the area of each washer in terms of the variable  $x$ .
- Washers perpendicular to a horizontal axis of rotation will have a  $dx$ -thickness.
- Identify the volume of each washer in terms of the variable  $x$ .
- Write an integral to sum the volume of all the washers.

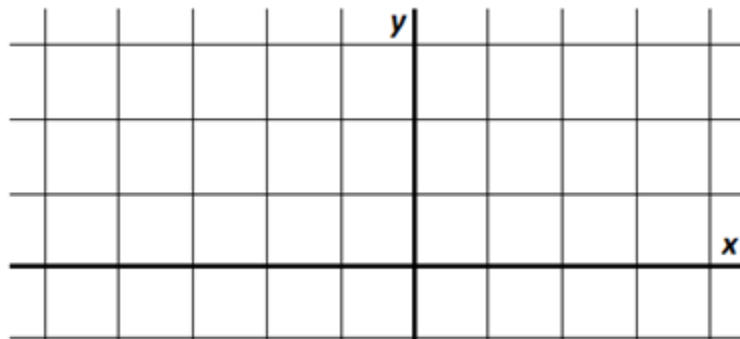


Show work here:

**Example 2:** The region between the curve  $y = \sqrt{x}$ ,  $0 \leq x \leq 4$  and the x-axis is revolved around the vertical line  $x = -1$  to generate a solid. Find its volume.

Follow the steps outlined above. Since the region is revolved about the vertical line-axis the washers are now perpendicular to the y-axis and they have a dy-thickness.

Show work here:



**Volume of Revolution by Washers**

**Volume of a solid obtained by revolving a bounded region about an axis of rotation:**

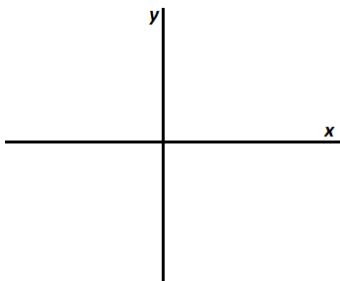
*about a horizontal axis      or      about a vertical axis*

$V = \pi \int_a^b \left( [R(x)]^2 - [r(x)]^2 \right) dx$ <p style="text-align: center;"><i>for <math>a \leq x \leq b</math></i></p>	$V = \pi \int_c^d \left( [R(y)]^2 - [r(y)]^2 \right) dy$ <p style="text-align: center;"><i>for <math>c \leq y \leq d</math></i></p>
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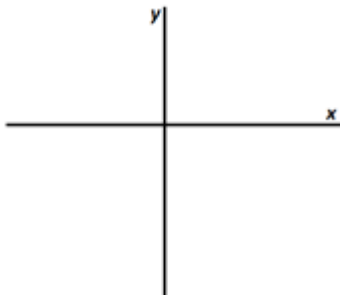
*Problems:*

Draw a picture and find the volume of the solid found by revolving the region about the stated axis.

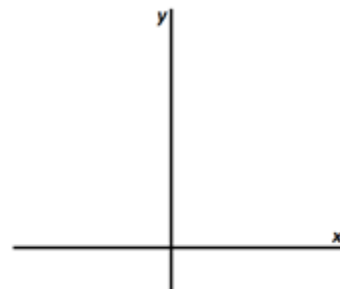
1.  $y = 2 - x^2$ ,  $y = 1$   
*x-axis*



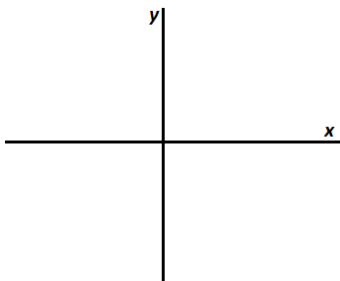
2.  $y = 2 - x^2$ ,  $y = 1$   
 $y = -3$



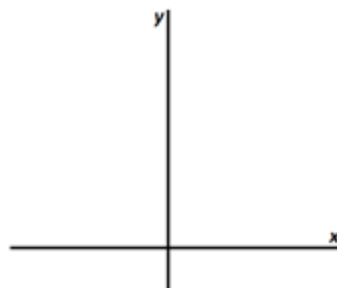
3.  $y = 2 - x^2$ ,  $y = 1$   
 $y = 3$



4.  $y = \frac{1}{x}$ ,  $y = \sqrt{x}$ ,  $x = 1$ ,  $x = 4$   
*x-axis*



5.  $y = \frac{1}{x}$ ,  $y = \sqrt{x}$ ,  $x = 1$ ,  $x = 4$   
 $y = 2$



6. The region in the first quadrant bounded by  $y = \sqrt{6x+4}$  &  $y = 2x$  when revolved about the *x-axis*.

7. The region bounded by  $y = x^3$ ,  $y = 0$  &  $x = 1$  when revolved about

a) the *y-axis*

b)  $x = -1$

c)  $x = 1$

## Volumes of Revolution (Disks/Washers)

*Work Smarter not Harder Calculator LAB (TI-83)*

It is common for solid objects to be produced by rotating a planar region about a line. The cross-sections of these solids are disks or very thin cylinders. We know that the volume of a cylinder is equal to the base area multiplied by height and the base area is the area of a circle so the volume is  $V = \pi r^2 h$ .

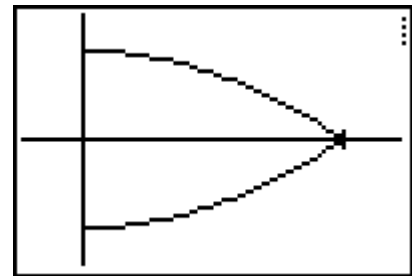
Using integration, the volume integral is  $V = \int \pi r^2 dx$  for rotations about a horizontal axis of rotation.  
 $V = \int \pi r^2 dy$  for rotations about a vertical axis of rotation.

One of the biggest difficulties in doing these problems is visualizing the solid. The program VOLUME can help you visualize the volumes of solids.

**Example 1:** Let  $R$  be the region in the first quadrant bounded by  $y = 4 - x^2$  and the coordinate axes. Find the volume of revolution when this region is rotated about the a)  $x$ -axis. b)  $y = -2$ .

Run the program VOLUME. Press [ENTER]. Select 1:ENTER CURVES. Select 1:F(x) to enter the given function in terms of  $t$ .

Enter CURVE 1 as  $4 - T^2$ . Enter CURVE 2 as 0. Enter Xmin as 0 and Xmax as 2. Enter Ymin as 0 and Ymax as 4. Press [ENTER]. Select 3:ENTER BOUNDS. Select 1:ENTER MANUALLY. Enter LOWER BOUND as 0 and UPPER BOUND as 2. Press [ENTER] to verify that you have correctly entered the bounds. The program then draws and shades the region that you will be rotating. Press [ENTER]. The program shows you the bounds and the area of the region.



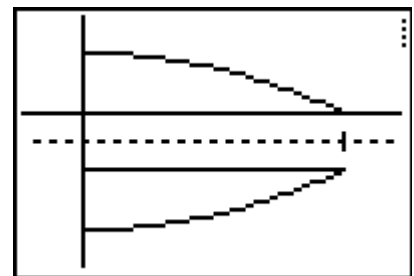
a) Rotate this region about the  $x$ -axis (the line  $y=0$ ). Select 4:AXIS ROTATION. Select 1:HORIZONTAL. Enter  $Y=0$ . Press [ENTER]. The program pictures your region and its reflection across the axis of rotation. Press [ENTER]. Select 5:DRAW RECTANGLE. CLEAR DRAW:NO. The cursor is at the left bound. Press [ENTER] and draw your first rectangle at the edge of the left bound. Once the rectangle has been drawn, press [ENTER] again to see the first disk. Press [ENTER] and choose 5:DRAW RECTANGLE. CLEAR DRAW:NO again. Move the cursor to the right to about 0.3 or 0.4. Draw another disk. Repeat the process until you have five or six disks across the region. Sketch the solid on the diagram.

In this example, the cross-sections are disks or tiny circular cylinders. What is the radius of each disk? \_\_\_\_\_

Write an integral expression for the volume of rotation. Volume= \_\_\_\_\_

Press [ENTER] and select 6:ANSWER. Compare the program answer with the calculator answer from evaluating this integral expression.

b) Now use the same region, but revolve it about a different horizontal line, the line  $y = -2$ . Press [ENTER] and choose 4:AXIS ROTATION. Select 1:HORIZONTAL. Enter  $Y = -2$ . Press [ENTER]. The program pictures the region and its reflection across the axis of rotation (the dotted line). Proceed as in the previous example. Sketch the solid on the diagram.



Note that there is a cylindrical hole in the middle of this solid. The cross-section consists of a circle with radius  $R = (4 - x^2) - (-2) = 6 - x^2$  from which a smaller circle with radius  $r = 2$  has been removed. The cross-sectional area is therefore,  
 $\pi R^2 - \pi r^2 = \pi (R^2 - r^2) = \pi [(6 - x^2)^2 - (2)^2]$ .

Write an integral expression for the volume of rotation: Volume= \_\_\_\_\_

Press [ENTER] and select 6:ANSWER. Compare the program answer with the calculator answer from evaluating this integral expression. What is the volume (rounded to the nearest thousandth)? \_\_\_\_\_

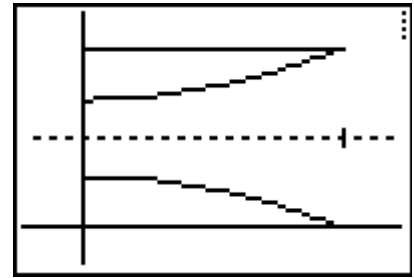
c) Use the same region, but revolve it about the horizontal line  $y=7$ . Proceed as in the previous examples.

Write an integral expression for the volume of rotation:

Volume= \_\_\_\_\_

Press [ENTER] and select 6:ANSWER. Compare the program answer with the calculator answer from evaluating this integral expression.

What is the volume (rounded to the nearest thousandth)? \_\_\_\_\_



**Example 2:** Examine a region  $R$  that is bounded by functions defined in terms of  $y$  and rotated about vertical axes.

Consider the region bounded by  $x = y^2$ ,  $x = 4$  and the horizontal lines  $y = 0$  and  $y = 2$ . As before select 1:ENTER

CURVES, but select 2:F(Y) to enter the given function in terms of  $t$ . Enter CURVE 1 as  $T^2$ . Enter CURVE2 as 4. Enter Xmin as 0. Enter Xmax as 4. Enter Ymin as 0 and Ymax as 2. Press [ENTER]. Select 3:ENTER BOUNDS. Select 1:ENTER MANUALLY. Enter LOWER BOUND as 0 and the UPPER BOUND as 2. Press [ENTER] to verify that you have correctly entered the bounds.

a) Rotate this region about the  $y$ -axis (the line  $x=0$ ). Select 4:AXIS ROTATION. Select 2:VERTICAL. Enter X= 0. Press [ENTER]. Proceed as in previous examples. Sketch the solid on the diagram.

Write an integral expression for the volume of rotation.

Volume= \_\_\_\_\_

What is the volume (rounded to the nearest thousandth)? \_\_\_\_\_

b) Now use the same region, but revolve it about the vertical line  $x = -2$ . Sketch the solid on the diagram.

Write an integral expression for the volume of rotation.

Volume= \_\_\_\_\_

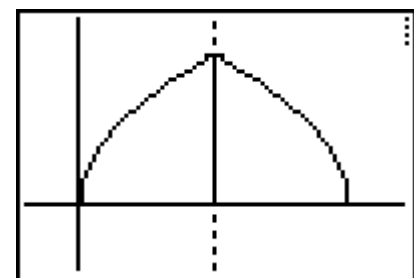
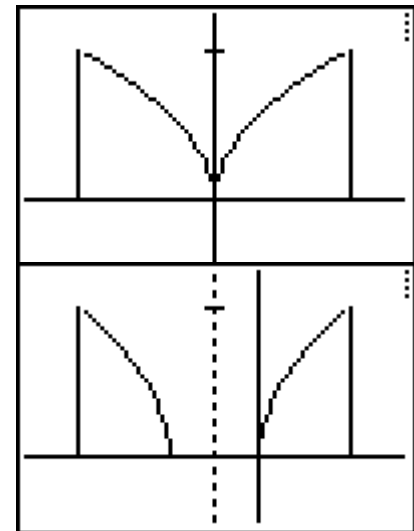
What is the volume (rounded to the nearest thousandth)? \_\_\_\_\_

c) Now use the same region, but revolve it about the vertical line  $x = 4$ . Sketch the solid on the diagram.

Write an integral expression for the volume of rotation.

Volume= \_\_\_\_\_

What is the volume (rounded to the nearest thousandth)? \_\_\_\_\_



**Example 3:** Draw the solid and write the integral expression for the volume of rotation if the region being rotated is bounded by the curves  $y = \sin x$  and

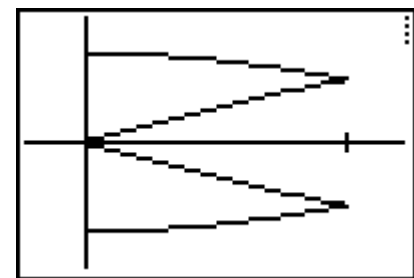
$y = \cos x$  on the interval  $\left[0, \frac{\pi}{4}\right]$ . Rotate the region about the  $x$ -axis.

Sketch the solid on the diagram.

Write an integral expression for the volume of rotation.

Volume= \_\_\_\_\_

What is the volume (rounded to the nearest thousandth)? \_\_\_\_\_



**HW DAY 119 Directions:** Sketch a neat & accurate graph. Reflect the bounded region. Draw a representative washer. Write the integral and evaluate.

- Find the volume of the solid formed by revolving the region bounded by the graph of  $f(x) = \sqrt{\sin x}$  and the  $x$ -axis from  $(0, \pi)$  about the  $x$ -axis. ANSWER:  $2\pi$
- Find the volume of the solid formed by revolving the region bounded by  $f(x) = 2 - x^2$  and  $y = 1$  that is revolved about the line  $y = 1$ . ANSWER:  $\frac{16\pi}{15}$
- Find the volume of the solid formed by revolving the region bounded by the graph of  $y = \sqrt{x}$  and  $y = x^2$  about the  $x$ -axis. ANSWER:  $\frac{3\pi}{10}$
- Graph the region bounded by the curve  $y = x^2$ , the  $x$ -axis, and the line  $x = 2$ . Then revolve it over each of the indicated axis of rotation, sketch a representative disk or washer, write the integral and evaluate to find the volume.

a. $x$ -axis	b. $y$ -axis	c. $x = 2$	d. $x = 3$	e. $y = -1$	f. $y = 5$
Answer 20.106	25.133	8.378	25.133	36.861	63.670

- Graph the region bounded by the curve  $y = x^2$  and  $y = \sqrt{x}$ , then revolve it over each of the indicated axis of rotation, sketch a representative disk or washer, write the integral and evaluate to find the volume.

a. $x$ -axis	b. $y$ -axis	c. $x = 1$	d. $y = 1$	e. $x = \frac{-1}{2}$
ANSWER 0.942	0.942	1.152	1.153	1.990

### Density Functions – an application of integrals

A *density function* gives the amount of something per unit of length, area, or volume, for example

- The density of a metal rod may be given in units of grams per centimeter.
- The density of the population of a city may be given in units of people per square mile.
- The density of a container of substance may be given in pounds per cubic foot.

The density can be used to find the amount. In each example, notice that the length, area, or volume of the region is multiplied by the density to find the amount.

Key Idea: slicing, based on density, rather than shape

Let's begin by giving some examples of the units of density --

A population density is measured in, say, people/mile along the edge of a road, people/unit area in a city, or bacteria/cm<sup>3</sup> in a test tube.

The density of a substance (air, wood, or metal) is the mass of a unit volume of the substance and is measured in units such as grams/cm<sup>3</sup>.

Suppose we wish to calculate the total mass or population but the density is not constant over the region, we must divide the region into smaller pieces in such a way that the density is approximately constant on each piece.

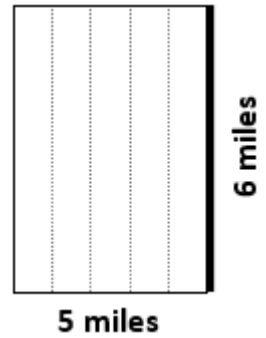
To calculate the total mass or total population where the density varies over the region,

- divide the region into small pieces of relatively constant density
- add up the contributions of all the pieces
- always use units in the setups.

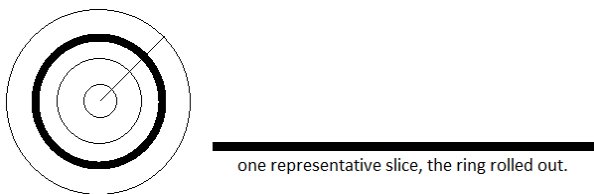
Example: Quadville is a city in the shape of a rectangle, five miles on one side and six miles on the other. A highway runs along the side that is six miles long. The population density  $x$  miles from the highway is given by  $p(x) = 20 - 4x$  thousand people per square mile. What is the approximate population of Quadville?

Solution:

1. Dimensional Analysis: What are the units involved in the scenario and what are the units of the answer?
2. Draw a diagram including the “slices”.
3. Determine the population of each representative slice:
4. Determine the total population:



Ex 1: The population density in Ringsburg is a function of the distance from the city center. Suppose that at  $r$  miles from the center, the density is given by the function  $p(r) = \sqrt{36 - r^2}$  in thousand persons/mile<sup>2</sup>. Ringsburg has a radius of 5 miles. Estimate the total population of Ringsburg.



Ex 2: The air density (in kg/m<sup>3</sup>)  $h$  meters above the earth’s surface is  $P = f(h)$ . Find the mass of a cylindrical column of air 12 meters in diameter and 25 kilometers high if  $f(h) = 1.28e^{-0.000124h}$ . Write the Riemann sum and definite integral and then evaluate. (Hint: Be careful with units.)



Ex 3: A city is built around a circular lake that has a radius of 1 mile. The population density of the city is  $f(r)$  people per square mile, where  $r$  is the distance from the center of the lake, in miles. Which of the following expressions gives the number of people who live within 1 mile of the lake?

(A)  $2\pi \int_0^1 r f(r) dr$       (B)  $2\pi \int_0^1 r (1 + f(r)) dr$

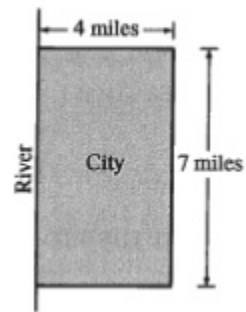
(C)  $2\pi \int_0^2 r (1 + f(r)) dr$       (D)  $2\pi \int_1^2 r f(r) dr$

(E)  $2\pi \int_1^2 r (1 + f(r)) dr$

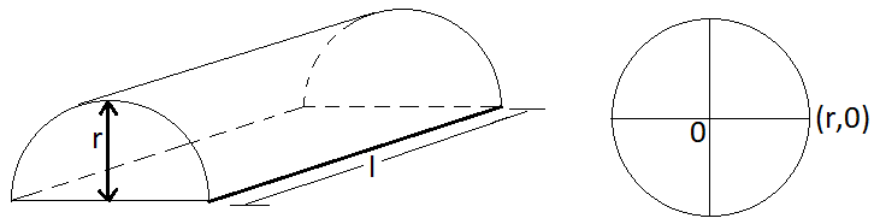
Ex 4: A city located beside a river has a rectangular boundary as shown in the figure above. The population density of the city at any point along a strip  $x$  miles from the river's edge is  $f(x)$  people per square mile. Which of the following expressions gives the population of the city?

(A)  $\int_0^4 f(x) dx$       (B)  $7 \int_0^4 f(x) dx$       (C)  $28 \int_0^4 f(x) dx$

(D)  $\int_0^7 f(x) dx$       (E)  $4 \int_0^7 f(x) dx$



Page 447 #19 A semi-cylindrical storage shed is shown below has radius  $r$  and length  $l$ .



- a. What is the volume of the shed?
  
- b. The shed is filled with sawdust whose density (mass/unit volume) at any point is proportional to the distance of that point from the floor. The constant of proportionality is  $k$ . Calculate the total mass of the sawdust.