DAY 71

§4.6 Related Rates—Student Notes

Any equation involving two or more variables that are differentiable functions of time, *t*, can be used to find an equation that relates their corresponding rates.

Related Rates

Identify the formula and then take the derivative with respect to time.

* Don't forget product rule. *
2, 4, 8, 14, 15

1. $A=\pi r^2$ circle area $\quad \frac{dA}{dt}=2\pi r\frac{dr}{dt}$

2. $V=\pi r^2 h$ cylinder volume $\quad \frac{dV}{dt}=\pi\left(2rh\frac{dr}{dt}+r^2\frac{dh}{dt}\right)$

3. $a^2+b^2=c^2$ Pythagorean Thm $\quad a\frac{da}{dt}+b\frac{db}{dt}=c\frac{dc}{dt}$

4. $S=2\pi rh+2\pi r^2$ surface area cylinder $\quad \frac{dS}{dt}=2\pi\left(\frac{dr}{dt}h+h\frac{dr}{dt}\right)+4\pi r\frac{dr}{dt}$

5. $A=s^2$ Area of square $\quad \frac{dA}{dt}=2s\frac{ds}{dt}$

6. $S=4\pi r^2$ surface area sphere $\quad \frac{dS}{dt}=8\pi r\frac{dr}{dt}$

7. $P=2l+2w$ perimeter rectangle $\quad \frac{dP}{dt}=2\frac{dl}{dt}+2\frac{dw}{dt}$

8. $V=\frac{1}{3}\pi r^2 h$ volume cone $\quad \frac{dV}{dt}=\frac{\pi}{3}\left(2rh\frac{dr}{dt}+r^2\frac{dh}{dt}\right)$

9. $V=s^3$ volume cube $\quad \frac{dV}{dt}=3s^2\frac{ds}{dt}$

10. $x^2+y^2=25$ circle w/ radius 5 $\quad 2x\frac{dx}{dt}+2y\frac{dy}{dt}=0$
    $x\frac{dx}{dt}+y\frac{dy}{dt}=0$

11. $C=2\pi r$ circumference circle. $\quad \frac{dC}{dt}=2\pi\frac{dr}{dt}$

12. $V=\frac{4}{3}\pi r^3$ volume sphere $\quad \frac{dV}{dt}=4\pi r^2\frac{dr}{dt}$

13. $P=4s$ perimeter square $\quad \frac{dP}{dt}=4\frac{ds}{dt}$

14. $A=lw$ Area of rectangle, p-gram. $\quad \frac{dA}{dt}=\left(\frac{dl}{dt}\cdot h+l\cdot\frac{dh}{dt}\right)$

15. $A=\frac{1}{2}bh$ Area of triangle $\quad \frac{dA}{dt}=\frac{1}{2}\left(\frac{db}{dt}h+b\frac{dh}{dt}\right)$

#1

DAY 71

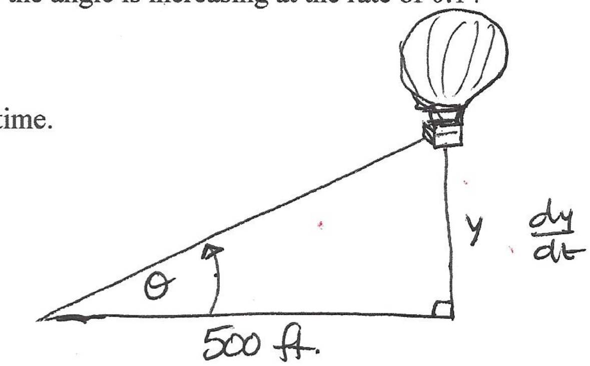
Example 1: A hot-air balloon rising straight up from a level field is tracked by a range finder 500 feet from the point of lift-off. At the moment the range finder's elevation angle is $\frac{\pi}{4}$, the angle is increasing at the rate of 0.14 rad/min. How fast is the balloon rising at that moment?

Step 1: Draw a picture and name the variables and constants. Use *t* for time.

$\theta = \frac{\pi}{4}$ $\quad \frac{d\theta}{dt} = 0.14 \frac{\text{rad}}{\text{min}}$ $\quad$ find $\frac{dy}{dt}$

$\downarrow$

$\therefore y = 500\text{ft}$



Step 2: Write down the numerical information (instantaneous rates of change, constants).

$\theta = \frac{\pi}{4}$ $\quad \frac{d\theta}{dt} = +0.14 \frac{\text{rad}}{\text{min}}$

$\therefore y = 500\text{ft.}$

Step 3: Write down what you are trying to find (usually a rate, expressed as a derivative).

$\frac{dy}{dt} = ?$

should be positive since balloon is rising.

Step 4: Write an equation that relates the variables. You may have to combine two or more equations to get a single equation that relates the variable whose rate you want to the variables whose rates you know.

$\tan\theta = \frac{y}{500}$

or $\quad 500\tan\theta = y$

Step 5: Use implicit differentiation with respect to *t* to find the derivatives of the equation in *Step 4*.

$500\sec^2\theta \frac{d\theta}{dt} = \frac{dy}{dt}$ $\quad$ or $\quad \sec^2\theta \frac{d\theta}{dt} = \left(\frac{1}{500}\right)\left(\frac{dy}{dt}\right)$

solve for $\frac{dy}{dt}$

Step 6: Now, and only now, substitute your instantaneous rates of change for the variables. The biggest mistake students make is substituting too early

$500\sec^2\left(\frac{\pi}{4}\right)(0.14) = \frac{dy}{dt}$

$\frac{dy}{dt} = 500(\sqrt{2})^2(0.14)$ $\quad \therefore \frac{dy}{dt} = 140 \frac{\text{ft.}}{\text{min.}}$

Step 7: Make sure your answer is reasonable and include units.

The balloon is rising at a rate of 140 ft/min when the angle of elevation from the range finder is $\theta = \frac{\pi}{4}$.

DAY

§4.6 Related Rates

Complete these Example Problems and HW problems in your notebook

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Example 2: Water runs into a conical tank at the rate of 9 ft^3/min. The tank stands point down and has a height of 10 ft and a base radius of 5 ft. How fast is the water level rising when the water is 6 ft deep?
(Answer: $\frac{1}{\pi}$ ft/min)
0.318

Example 3: A man 6 ft tall walks at a rate of 5 ft/sec toward a street light that is 16 ft above the ground. At what rate is the length of his shadow changing when he is 10 ft from the base of the light? **(Answer: –3 ft/sec)**

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Example 4: A pebble is dropped into a calm pond, causing ripples in the form of concentric circles. The radius, r, of the outer ripple is increasing at a constant rate of 1 ft/sec. When this radius is 4 ft, at what rate is the total area of the disturbed water increasing? **(Answer: 8π ft^2/sec)**

Example 5: Gravel is falling in a conical pile at the rate of 100 ft^3/min. Find the rate of change of the height of the pile when the height is 10 ft. Assume that the coarseness of the gravel is such that the radius of the cone is always equal to its height. **(Answer: $\frac{1}{\pi} \approx 0.318$ ft/min)**

HW: Related Rates $\boxed{\text{DAY 71-72 HW}}$

HW #1: The width of a rectangle is increasing at a rate of 2 cm/sec and its length is increasing at a rate of 3 cm/sec. At what rate is the area of the rectangle increasing when its width is 4 cm and length is 5 cm? At what rate is the length of the diagonal of the rectangle increasing? At what rate is the perimeter of the rectangle increasing?

HW #2: A spherical ball 8 inches in diameter is coated with a layer of ice of uniform thickness. If the ice melts at a rate of 10 in^3/min, how fast is the thickness of the ice decreasing when it is 2 inches thick? How fast is the outer surface area of the ice decreasing at this time?

HW #3: A baseball diamond is a square 90 feet on one side. A runner travels from home plate to first base at 20 ft/sec. How fast is the runner's distance from second base changing when the runner is halfway to first base?

Day 73 in class.

HW #4: A rocket rises vertically from a point on the ground that is 100 m from an observer at ground level. The observer notes that the angle of elevation is increasing at a rate of 12 degrees per second when the angle of elevation is 60 degrees. Find the speed of the rocket at that instant. *convert to radians
*convert to radians

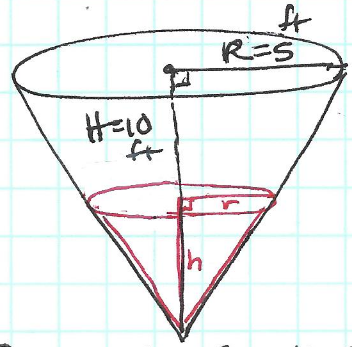
HW #5: If $x^2 + y^2 = 25$ and $\frac{dy}{dt} = 6$ cm/sec, find $\frac{dx}{dt}$ when $y = 4$ cm.

HW #6: Sand is being dumped on a pile in such a way that it always forms a cone whose radius equals its height. If the sand is being dumped at a rate of 10 ft^3/min, at what rate is the height of the pile increasing when there is 1000 ft^3 of sand on the pile?

§4.6 EXAMPLE RELATED RATES PROBLEMS.

DAY 71 NOTES

#2)



$$\frac{dV}{dt} = 9\,\frac{ft^3}{min} \qquad V = \frac{1}{3}\pi r^2 h$$

$$\frac{dh}{dt} = ?$$

When $h = 6ft$

SIMILAR TRIANGLES

$$\frac{R}{H} = \frac{r}{h}$$

$$\frac{5}{10} = \frac{r}{h}$$

$$\frac{1}{2}h = r$$

$h=6 \therefore r=3$

$$\therefore \frac{1}{2}\frac{dh}{dt} = \frac{dr}{dt}$$ use this one

or $\frac{dh}{dt} = 2\frac{dr}{dt}$

$$\frac{dV}{dt} = \frac{\pi}{3}\left(2rh\frac{dr}{dt} + r^2\frac{dh}{dt}\right)$$

$$\frac{dV}{dt} = \frac{\pi}{3}\left(2rh\left(\frac{1}{2}\frac{dh}{dt}\right) + r^2\frac{dh}{dt}\right)$$

$$\frac{dV}{dt} = \frac{\pi}{3}\left((rh + r^2)\frac{dh}{dt}\right)$$

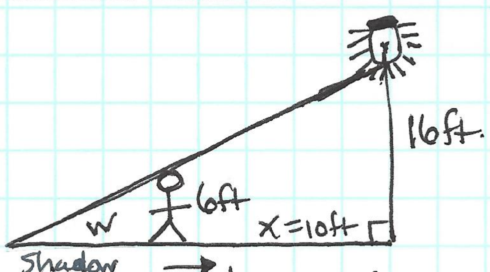
$$\frac{dh}{dt} = \frac{3\cdot\left(\frac{dV}{dt}\right)}{\pi\cdot(rh + r^2)}$$

$$\frac{dh}{dt} = \frac{3\,(9)}{\pi\,(3(6) + 3^2)} = \frac{\cancel{27}}{\pi\cancel{27}}$$

$$\frac{dh}{dt} = \frac{\cancel{3}\cdot\cancel{9}}{\pi(\cancel{3})(\cancel{6+3})} = \frac{1}{\pi}$$

ATQ: The water level is rising at a rate of $\frac{1}{\pi}\frac{ft}{min}$ when the water is 6ft deep.

#3)



$$\frac{dw}{dt} = ? \qquad \frac{dx}{dt} = -5\,\frac{ft}{sec}$$

when $x = 10ft$

$$\frac{6}{w} = \frac{16}{w+10}$$

$$6w + 60 = 16w$$

$$60 = 10w$$

$$w = 6\ ft.$$

$$\frac{w}{6} = \frac{w+x}{16}$$

$$16w = 6w + 6x$$

$$10w = 6x$$

$$w = \frac{3}{5}x$$

$$\frac{dw}{dt} = \frac{3}{5}\frac{dx}{dt}$$

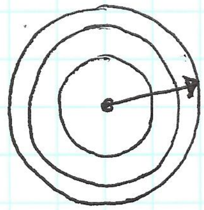
$$\frac{dw}{dt} = \frac{3}{5}\left(-5\,\frac{ft}{sec}\right)$$

$$\boxed{\frac{dw}{dt} = -3\,\frac{ft}{sec.}}$$

ATQ: The length of the mens shadow is decreasing at a rate of 3 ft/sec when the man is 10 ft from the lampost.

DAY 72 NOTES EXAMPLES p. 20 #4–5

#4)



$\frac{dr}{dt} = 1 \frac{ft}{sec.}$ $r = 4 ft.$

$A = \pi r^2$

$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$

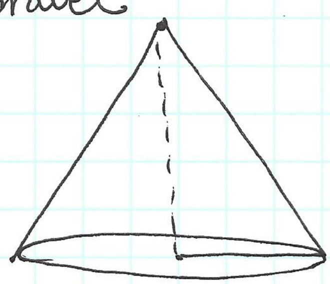
$\frac{dA}{dt} = 2\pi(4)(1)$

$\frac{dA}{dt} = 8\pi \frac{ft^2}{sec}$

$\frac{dr}{dt}$ is positive b/c Area of circle is increasing.

ATQ. The area of the disturbed water is increasing at a rate of $8\pi \frac{ft^2}{sec}$ when the radius is 4ft.

#5) Gravel



$\frac{dV}{dt} = 100 \frac{ft^3}{min}$ Find $\frac{dh}{dt} = ? \frac{ft}{min}$

& $r = h$ when $h = 10 ft.$

$\therefore \frac{dr}{dt} = \frac{dh}{dt}$

$$V = \frac{1}{3}\pi r^2 h$$

$$\frac{dV}{dt} = \frac{\pi}{3}\left(2rh\frac{dr}{dt} + r^2\frac{dh}{dt}\right)$$

$$\frac{dV}{dt} = \frac{\pi}{3}\left(2h^2\frac{dh}{dt} + h^2\frac{dh}{dt}\right)$$

$$\frac{dV}{dt} = \frac{\pi}{3}(3h^2)\frac{dh}{dt}$$

$$\frac{dV}{dt} = \pi(h^2)\frac{dh}{dt}$$

$$\frac{dh}{dt} = \left(\frac{1}{\pi h^2}\right)\left(\frac{dV}{dt}\right) = \left(\frac{1}{\pi 10^2}\right)(100) = \frac{1}{\pi}\frac{ft}{min}$$

The height of the gravel in the conical pile is increasing at a rate of $\frac{1}{\pi} \approx 0.3183 \frac{ft}{min}$ at the moment the height is 10ft.