DAY 68

§4.3 Student Notes -- Optimization & Modeling

Example 1 *Finding the maximum volume.* An open box having a square base is to be constructed from 108 in^2 of material. What dimensions will produce a box with maximum volume?

Step 1 State what quantity you are trying to optimize. MAXIMIZE Volume

Step 2 Make a sketch if one is not provided. Label it.

Step 3 Write a formula for the quantity you want to optimize. $V = x^2 \cdot h$

Step 4 The formula must be written in terms of a single variable. By substituting from a "secondary equation" & rewrite the formula. (This step might not be needed!)

$SA = 108 = 4xh + x^2$

$\therefore h = \dfrac{108 - x^2}{4x}$

$V = x^2 \left(\dfrac{108 - x^2}{4x}\right)$

NOTE: Domain $x > 0$ $\longrightarrow$ $V = \frac{1}{4}x\,(108 - x^2)$

Step 5 Take the derivative of the equation from step 4.

$\dfrac{dV}{dx} = \frac{1}{4}\left[1(108 - x^2) + x(-2x)\right]$

$\dfrac{dV}{dx} = \frac{1}{4}(108 - x^2 - 2x^2)$

$\dfrac{dV}{dx} = \frac{1}{4}(108 - 3x^2)$

Step 6 Find the critical numbers. $f' = 0$ or f' und.
Set the derivative equal to zero and solve.
Throw out those that fall outside of the domain.
State why you are throwing them out.

$\dfrac{dV}{dx} = 0 = 108 - 3x^2 \;\therefore\; 3x^2 = 108$

$x^2 = 36$

$x = \pm 6$

B/c domain $x > 0$ exclude $x = -6$.

Step 7 Establish whether or not the critical number(s) correspond to local extrema and be sure to identify whether you have a max or min. (You DON'T want to minimize something you are trying to maximize or vise-versa!)

$\dfrac{dV}{dx}$: $-$ 0 $+$ 0 $-$

-6 $\quad$ 0 $\quad$ 6

MIN $\quad$ MAX.

$x = 6$ results in Max. Volume since $\dfrac{dV}{dx}$ changes signs from $+$ to $-$

Step 8 Find the value of the quantity to be optimized at each of the relevant critical numbers and answer the original problem in words with units.

$V(6) = 108 \text{ in}^3$

$x = 6$

$h = 3$

DIMENSIONS: $\boxed{6 \times 6 \times 3} \longrightarrow V = 108 \text{ in}^3$

$SA = 36 + 4(6)(3)$

$SA = 36 + 72$

$SA = 108 \text{ in}^2$

DAY

4.3 MODELING & OPTIMIZATION

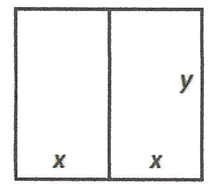
§4.3 Optimization Problems: **Solve these on your own paper in your notebook.**

68

#2:
Find two positive numbers that minimize the sum of twice the first number and the second, if the product of the two numbers is 288.

#3:
What are the dimensions of a closed cylindrical can that can hold 40 cubic inches of liquid and uses the least amount of material?

#4:
A rancher has 200 feet of fencing to enclose two rectangular fields as shown in the diagram. What are the dimensions of the field that maximize the area?



69

#5:
Find two positive integers whose sum is 20 and whose product is as large as possible.

#6:
An oil can is to be made in the form of a right circular cylinder to contain 1 quart of oil. What dimensions of the can will require the least amount of material? (1 quart liquid = 57.75 cubic inches)

70

#7:
If you have 100 feet of fencing and you want to enclose a rectangular area adjacent to the long side of the barn, what is the largest area you can enclose? 2nd derivative Test

#8:
The product of two positive numbers is 192. Find the two numbers such that the sum of the first and three times the second is a minimum. 2nd derivative Test

#9:
A right circular cylinder is to be designed to hold 12 fluid ounces using the least amount of material. Find the dimensions of the cylinder. (1 fluid ounce = 1.80469 cubic inches) 2nd Derivative Test.

#10:
A glass fish tank is to be constructed to hold 72 cubic feet of water. It's base and sides are to be rectangular. The top of the tank is open, of course. The width is 5 ft but the length and depth are variable. Building the tank costs \$10/sqft for the base and \$5/sqft for the lateral sides. Find cost and the dimensions of the tank that minimize the cost. 2nd Derivative Test.

69 #11:
Find the point on the parabola $x = -y^2$ that is closest to the point (0,-3).

70 #12:
Find the area and the dimensions of a rectangle bounded by the function $y = \cos(x)$ and the x-axis whose area is maximized.

69 #13:
Find the area and dimensions of a rectangle bounded by the curve $y = -x^2 + 4$ and the x-axis and y-axis whose area is maximized.

$\boxed{\text{DAY 68}}$ NOTES ANSWERS. p. 17 #2-3-4

② two positive numbers: $x > 0,\ y > 0$

MINIMIZE sum: $S = 2x + y$

Product $x \cdot y = 288$

$\therefore y = \dfrac{288}{x}$

$\text{Sum} = 2x + \dfrac{288}{x}$

$\dfrac{dS}{dx} = 2 - \dfrac{288}{x^2}$

$= \dfrac{2x^2 - 288}{x^2}$

$\dfrac{dS}{dx} = \dfrac{2(x-12)(x+12)}{x^2}$

Related Poly + 4th deg.

$\dfrac{dS}{dx} = 0$ $\qquad$ $\dfrac{dS}{dx}$ und.

$x = \pm 12$ $\qquad$ $x = 0$

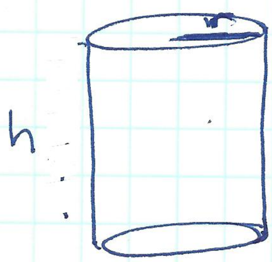
$\dfrac{dS}{dx}$: $+$ 0 $-$ Ø $-$ 0 $+$

$\nearrow$ -12 MAX $\searrow$ 0 TERR $\searrow$ 12 MIN $\nearrow$

JUSTIFY: Sum is minimized when $x = 12$ b/c $\dfrac{dS}{dx}$ changes signs $-$ to $+$

ATQ: $\underline{x = 12}$ $\quad$ $\underline{y = 24}$ $\quad$ Sum $= 48$.

③ Closed cylindrical can.



$V = 40\,in^3 = \pi r^2 h$

$\therefore h = \dfrac{40}{\pi r^2}$ $\longrightarrow$

MINIMIZE $SA = 2\pi r^2 + 2\pi r h$

$SA = 2\pi r^2 + 2\pi r\left(\dfrac{40}{\pi r^2}\right)$

$SA = 2\pi r^2 + \dfrac{80}{r}$

$\dfrac{dS}{dr} = 4\pi r + \dfrac{-80}{r^2} = \dfrac{4\pi r^3 - 80}{r^2}$

$\dfrac{dS}{dr} = \dfrac{4\pi\left(r^3 - \frac{20}{\pi}\right)}{r^2}$

$\dfrac{dS}{dr} = \dfrac{4\pi\left(r - \sqrt[3]{\frac{20}{\pi}}\right)\left(r^2 + \sqrt[3]{\frac{20}{\pi}}\,r + \left(\frac{20}{\pi}\right)^{2/3}\right)}{r^2}$

related poly + cubic

$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$

$\dfrac{dS}{dr} = 0$ $\qquad$ $\dfrac{dS}{dr}$ und.

$r = \sqrt[3]{\dfrac{20}{\pi}}$ $\qquad$ $r = 0$

$\dfrac{dS}{dr}$: $-$ Ø $-$ 0 $+$

$\searrow$ 0 $\searrow$ $\sqrt[3]{\dfrac{20}{\pi}}$ $\nearrow$

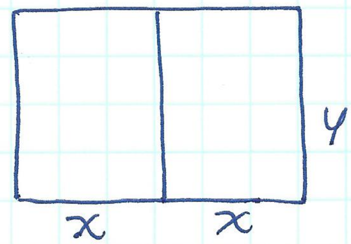
JUSTIFY: Surface area is minimized when $r = \sqrt[3]{\dfrac{20}{\pi}}$ b/c $\dfrac{dS}{dr}$ changes signs from $-$ to $+$.

ATQ: $r = \sqrt[3]{\dfrac{20}{\pi}}$ $\qquad$ $h = \dfrac{40}{\pi\left(\frac{20}{\pi}\right)^{2/3}} = \dfrac{40}{\sqrt[3]{400\pi}}$ $\qquad$ $\underline{\text{MIN. } SA = 64.7472\ in^2}$

$\underline{r = 1.8533\ in}$ $\qquad$ $\underline{h \approx 3.7067\ in}$

$\boxed{\text{DAY 68}}$ NOTES ANSWERS p. 17 #4

(4) Rectangular Fields.



$P = 200\text{ ft} = 4x + 3y$

MAXIMIZE AREA $= 2xy$

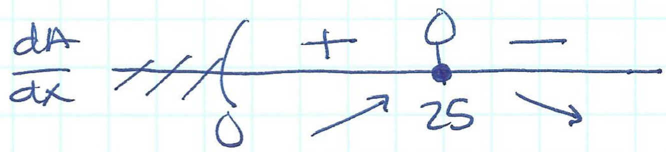
$\frac{200-4x}{3} = y \Rightarrow A = 2x\left(\frac{200-4x}{3}\right) = \frac{8}{3}(x)(50-x)$

Domain $x > 0$
$y > 0$

$\frac{dA}{dx} = \frac{8}{3}\left((50-x) + (x)(-1)\right)$

$\frac{dA}{dx} = \frac{8}{3}\left(50-2x\right) = \frac{16}{3}(25-x)$

$\frac{dA}{dx} = 0 \quad x = 25$



JUSTIFY

when $x = 25$ ft area is maximized b/c $\frac{dA}{dx}$ changes signs + to −.

$y = \frac{100}{3}$ ft

$A = \frac{5000}{3}\text{ ft}^2$

DAY 69 NOTES ANSWERS p. 17 # 5, 6, 11, 13

(5) two positive integers $x>0$, $y>0$

$\text{sum} = x+y = 20$

$y = 20-x$

MAX Product $= (x)(y)$

$= (x)(20-x)$

$\frac{dP}{dx} = (x)(-1) + (1)(20-x)$

$= -x + 20 - x$

$\frac{dP}{dx} = 20-2x$

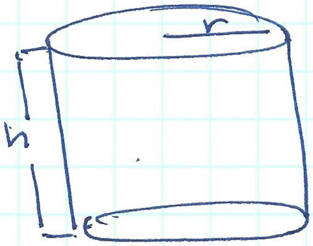
$\frac{dP}{dx} = 0 \quad x = 10$

$\frac{dP}{dx}$: $+$ | 0 at 10 | $-$; ↗ 10 ↘ MAX.

JUSTIFY: product is maximized when $x=10$ & $y=10$ b/c $\frac{dP}{dx}$ changes signs from $+$ to $-$.

ATQ: $x=10$, $y=10$, product $=100$

(6) Oil can MINIMIZE Surface Area $SA = 2\pi r^2 + 2\pi r h$



$1 \text{ quart} = 57.75 \text{ in}^3$

$V = \pi r^2 h = 57.75$

$h = \frac{57.75}{\pi r^2}$

domain: $r > 0$

$SA = 2\pi r^2 + 2\pi r\left(\frac{57.75}{\pi r^2}\right)$

$SA = 2\pi r^2 + \frac{115.50}{r}$

$\frac{dS}{dr} = 4\pi r - \frac{115.50}{r^2}$

$\frac{dS}{dr} = \frac{4\pi r^3 - 115.5}{r^2}$

$\frac{dS}{dr} = 0 \quad 4\pi r^3 - 115.5 = 0$

$4\pi r^3 = 115.5$

$r^3 = \frac{115.5}{4\pi}$

$r = \sqrt[3]{\frac{115.5}{4\pi}}$

$\frac{dS}{dr}$ und. $r = 0$.

$\frac{dS}{dr}$: $-$ | $\cancel{0}$ at 0 | $-$ | 0 at $\sqrt[3]{\frac{115.5}{4\pi}} \approx 2.0947$ | $+$; ↘ ↗

JUSTIFY: Surface area is minimized when $r = \sqrt[3]{\frac{115.5}{4\pi}} \approx 2.0947 \text{ in}$ b/c $\frac{dS}{dr}$ changes signs $-$ to $+$.

ATQ. $r = \sqrt[3]{\frac{115.5}{4\pi}} \approx 2.0947 \text{ in}$

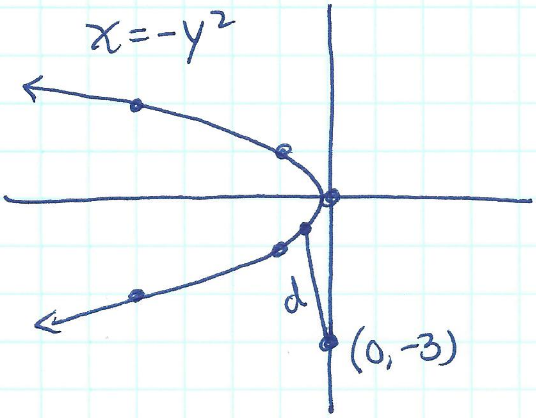
$h = \frac{57.75}{\pi\left(\sqrt[3]{\frac{115.5}{4\pi}}\right)^2} \approx 4.1894 \text{ in}$

$SA = 82.7083 \text{ in}^2$

$\boxed{\text{DAY 69}}$ NOTES ANSWERS p.17 # 11, 13

(11) Minimize distance btwn $(0,-3)$ & point (x,y) on $x=-y^2$

$x=-y^2$



$$d=\sqrt{(x-0)^2+(y+3)^2}$$

Minimize d^2 for easier derivative

$$d^2 = x^2+(y+3)^2 \quad \& \quad x=-y^2$$

$$d^2 = y^4+(y+3)^2$$

Min d^2 $\therefore$ $\dfrac{d(d^2)}{dy} = 4y^3+2(y+3)^1$

$$= 4y^3+2y+6 = 0$$

By inspection $y=-1$ $\quad 4(-1)^3+2(-1)+6=0$

or solve on graphing calculator $y=-1$.

$\dfrac{dd^2}{dy}$ $\quad - \quad 0 \quad +$; $\searrow \; -1 \; \nearrow$

JUSTIFY: when $y=-1$ the distance is minimized b/c $\dfrac{d(d^2)}{dy}$ changes signs from $\ominus$ to $\oplus$.

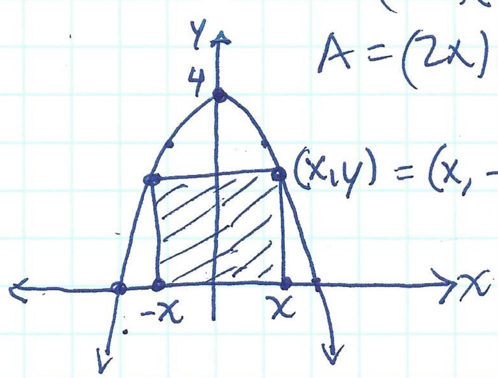
ATQ. Point closest to $(0,-3)$ on $x=-y^2$ is $\underline{(-1,-1)}$.

(13) $y=-x^2+4$

MAXIMIZE AREA

$$A=(2x)(y)$$

$$A=(2x)(-x^2+4)$$



$$\frac{dA}{dx} = (2)(-x^2+4)+(2x)(-2x)$$

$$= -2x^2+8-4x^2$$

$$= -6x^2+8$$

$$= -2(3x^2-4)$$

$$= -2(\sqrt{3}x-2)(\sqrt{3}x+2)$$

$\dfrac{dA}{dx}=0 \quad x=\pm\dfrac{2}{\sqrt{3}}$

$\dfrac{dA}{dx}$ $\quad - \quad 0 \quad + \quad 0 \quad -$; $\searrow \; -\dfrac{2}{\sqrt{3}} \; \nearrow \; +\dfrac{2}{\sqrt{3}} \; \searrow$

ATQ. $x=\dfrac{2}{\sqrt{3}}$ (1.1547 u) $\quad -x = -\dfrac{2}{\sqrt{3}} \approx -1.1547$ u

$y = \dfrac{4}{3}+4 = \dfrac{16}{3} \approx 5.333$ u

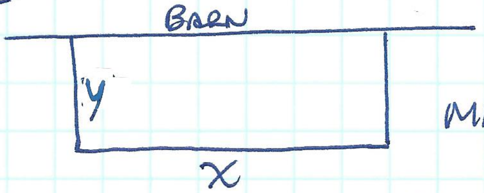
Area $= \left(\dfrac{4}{\sqrt{3}}\right)\left(\dfrac{16}{3}\right) = \dfrac{64}{3\sqrt{3}} \approx 12.3168\ u^2$

JUSTIFY: Area is maximized when $x=\dfrac{2}{\sqrt{3}}$ b/c $\dfrac{dA}{dx}$ changes signs from $\oplus$ to $\ominus$.

$\boxed{\text{DAY 70}}$ NOTES ANSWERS p.17 #7, 8, 9, 10, 12

(7) Perimeter $= 100$ ft $= x + 2y \longrightarrow y = \dfrac{100-x}{2}$

Domain $x > 0$



$\text{Area} = x \cdot y = x\left(\dfrac{100-x}{2}\right)$

MAXIMIZE AREA

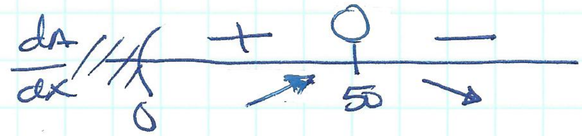
$= \frac{1}{2}(x)\left(50 - \frac{1}{2}x\right)$

$\dfrac{dA}{dx} = \left(\frac{1}{2}\right)\left(50 - \frac{1}{2}x\right) + \frac{1}{2}x\left(-\frac{1}{2}\right)$

$\dfrac{dA}{dx} = 25 - \frac{1}{4}x - \frac{1}{4}x = 25 - \frac{1}{2}x$

$\boxed{\dfrac{d^2A}{dx^2} = -\frac{1}{2}}$

$\dfrac{dA}{dx} = 0 \quad x = 50$



$\boxed{\text{JUSTIFY}} \rightarrow$ Area is maximized when $x = 50$ ft b/c $\dfrac{d^2A}{dx^2} < 0$ so Area is concave down and has $\underline{\text{MAX}}$ at $x = 50$. } 2nd Derivative Test.

ATQ. $x = 50$ ft $y = 25$ ft Area $= 1250$ ft^2

(8) Product $(x)(y) = 192 \rightarrow y = \dfrac{192}{x}$

domain $x > 0 \quad y > 0$

$\underline{\underline{\text{MIN}}}$ Sum $= 3x + y$

$\text{Sum} = 3x + \dfrac{192}{x}$

$\dfrac{dS}{dx} = 3 - \dfrac{192}{x^2}$

$= \dfrac{3x^2 - 192}{x^2}$

$= \dfrac{3(x^2 - 64)}{x^2}$

$= \dfrac{3(x-8)(x+8)}{x^2}$

$\dfrac{d^2S}{dx^2} = \dfrac{2(192)}{x^3} = \dfrac{384}{x^3}$

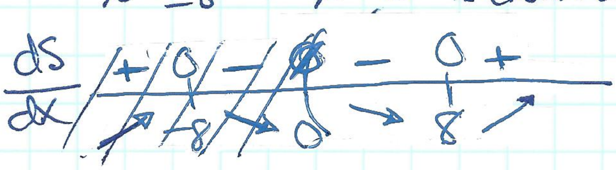
$\left.\dfrac{d^2S}{dx^2}\right|_{x=8} > 0 \therefore$ Sum is concave up & $S(8)$ is a minimum by 2nd Deriv Test

ATQ: $x = 8$
$y = 24$
Sum $= 48$.

$\dfrac{dS}{dx} = 0 \qquad \dfrac{dS}{dx}$ und

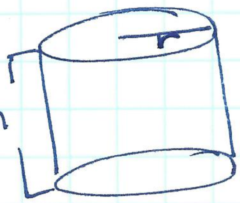
Domain $x > 0$

$x = \pm 8 \qquad x = 0 \quad \therefore$ exclude $x = 0$, $x = -8$



$\boxed{\text{DAY 70}}$ p. 17 #9, 10, 12 NOTES ANSWERS.

9



$$(12)(1.80469)$$
$$V = 21.65628 \text{ in}^3 = \pi r^2 h$$
$$h = \frac{(12)(1.08469)}{\pi r^2}$$

MINIMIZE SA.

$$SA = 2\pi r h + 2\pi r^2$$
$$SA = 24\pi r\left(\frac{1.80469}{\pi r^2}\right) + 2\pi r^2$$
$$SA = \frac{43.31256}{r} + 2\pi r^2$$

* Can is 12 fluid ounces.

$$\frac{d(SA)}{dr} = -\frac{43.31256}{r^2} + 4\pi r$$
$$\frac{d(SA)}{dr} = \frac{-43.31256 + 4\pi r^3}{r^2}$$

$$\frac{d(SA)}{dr} = 0 \qquad r^3 = \frac{43.31256}{4\pi}$$
$$r = \sqrt[3]{\frac{43.31256}{4\pi}}$$
$$r \approx 1.510548463$$

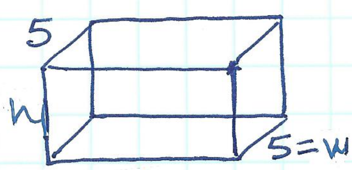
$$\frac{d^2(SA)}{dr^2} = \frac{86.62512}{r^3} + 4\pi$$

$$\left.\frac{d^2(SA)}{dr^2}\right|_{r=1.5105} = 37.699 > 0$$

$\therefore$ SA is minimized
when $r = 1.5105$
b/c SA is concave up
at this critical point for r.
by <u>2nd Derivative Test</u>.

10 Glass Fish Tank. $72 \text{ ft}^3 = V = l \cdot w \cdot h$

$$V = 5lh = 72$$
$$h = \frac{72}{5l}$$



Domain: $h > 0,\ l > 0$

MINIMIZE COST!

$$\$10(5l) + \$5(2l+10)(h) = \text{Cost}.$$
$$50l + 10h(l+5) = \text{Cost}.$$
$$\text{Cost} = 50l + 10\left(\frac{72}{5l}\right)(l+5)$$
$$\text{Cost} = 50l + \left(\frac{144}{l}\right)(l+5)$$
$$\text{Cost} = 50l + 144 + \frac{720}{l}$$

$\boxed{\text{JUSTIFY}}$ $\left.\frac{d^2C}{dl^2}\right|_{l=\sqrt{\frac{72}{5}}} > 0 \therefore$

Cost function is concave up at critical point $l = \sqrt{\frac{72}{5}}$
$\therefore$ Minimizing cost by 2nd Der. Test.

~~ATQ~~ Cost = \$523.47
$l = 3.7947$ ft. $h = 3.7947$ ft

$$\frac{dC}{dl} = 50 - \frac{720}{l^2} = \frac{50l^2 - 720}{l^2}$$

$$\frac{dC}{dl} = 0 \qquad l = \pm\sqrt{\frac{720}{50}} = \pm\sqrt{\frac{72}{5}}$$

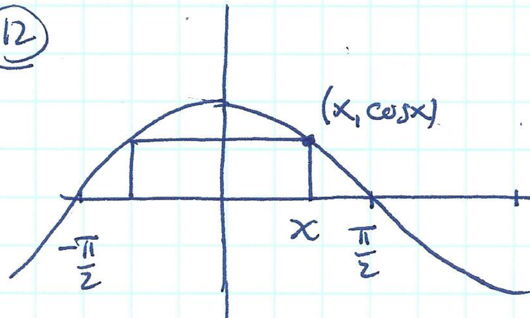
$\frac{dC}{dl}$ und $l = 0$

$\frac{dC}{dl}$: + 0 − ∅ − 0 +
$\nearrow -\sqrt{\frac{72}{5}} \searrow 0 \searrow \sqrt{\frac{72}{5}} \nearrow$

$$\left.\frac{d^2C}{dl^2} = +\frac{1440}{l^3}\right|_{l=\sqrt{\frac{72}{5}}} = 26.352 > 0$$

DAY 70 p.17 #12 NOTES ANSWERS

(12)



domain $-\frac{\pi}{2} < x < \frac{\pi}{2}$

MAX of rectangle.

$\text{Area} = (2x)(\cos x)$

$$\frac{dA}{dx} = 2\cos x - 2x\sin x$$

$$2\underbrace{(\cos x - x\sin x) = 0}_{\text{Solve on graphing calculator.}}$$

$x = \pm 0.86033359$

$\frac{dA}{dx}$: $-$ ↘ 0 at -0.860 MIN. $+$ ↗ 0 at $+0.860$ MAX $-$ ↘

JUSTIFY

$\frac{dA}{dx}$ changes signs $+$ to $-$ at $x = 0.860$ verifying a maximum area when $x = 0.860$ & height $= 0.652$ & Area $= 1.122$