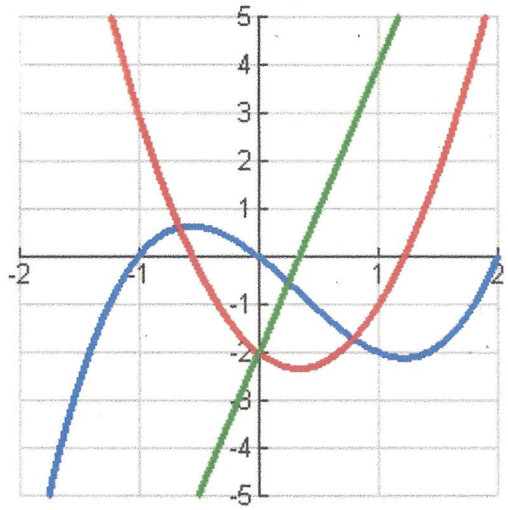
ANSWERS

§4.1 & §4.2—Day 2—Student Notes—Using the First and Second Derivatives

DAY 60



$f(x) = x^3 - x^2 - 2x$

$f'(x) =$ $3x^2-2x-2$

$f''(x) =$ $6x-2$

f'	f''
$f'>0$ 1. Positive: f is increasing	$f''>0$ 1. Positive: f' is increasing f is concave up
$f'<0$ 2. Negative: f is decreasing	$f''<0$ 2. Negative: f' is decreasing f is concave down
$f'=0$ 3. Intercepts: extrema on f rel max rel min	$f''=0$ 3. Intercepts: extrema on f' inflection pts on f
f' has 4. Max/Min: inflection points on f	
f' slope $= f''>0$ 5. Slope is positive: conc up on f positive on $f''>0$	
f' slope $= f''<0$ 6. Slope is negative: conc down on f negative on $f''<0$	

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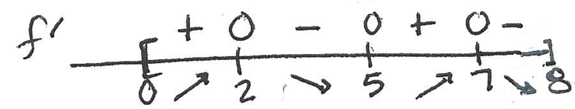
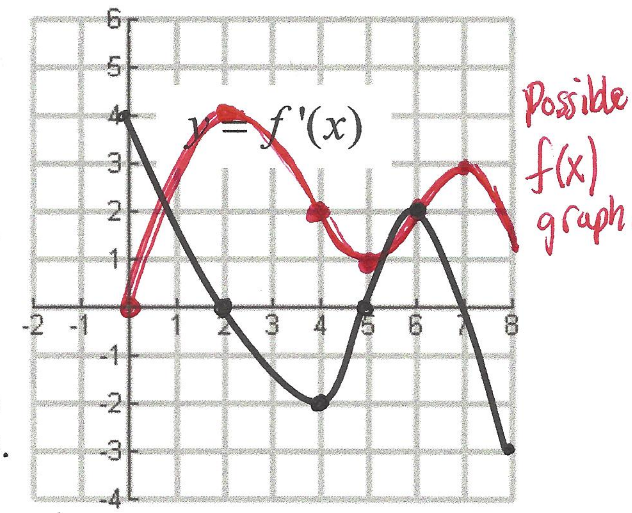


Example 1: The graph of the derivative f' of a continuous function f is shown on the interval [0,8]:



a. On what intervals is f increasing or decreasing?

↗ on (0,2) (5,7) b/c $f'(x)>0$ ↘ on (2,5) (7,8) b/c $f'(x)<0$

b. At what values of x does f have a local maximum or minimum?

MAX @ $x=2,7$ b/c f' changes ⊕ to ⊖ MIN @ $x=5$ b/c f' changes ⊖ to ⊕.

c. On what intervals is f concave upward or downward?

∪ on (4,6) b/c $f''>0$. ∩ on (0,4), (6,8) b/c $f''<0$.

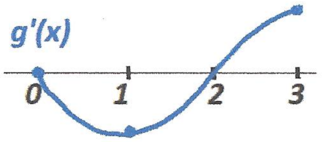
d. State the $x-$coordinates of the points of inflection.

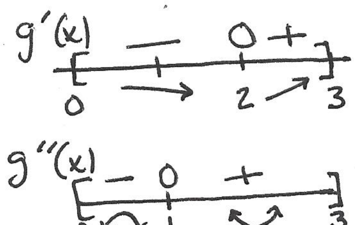
Inf. Pts @ $x=4$ b/c f'' changes sign ⊖ to ⊕
@ $x=6$ b/c f'' changes sign ⊕ to ⊖.

e. Assume that $\boxed{f(0)=0}$, sketch the graph of f.

General shape inc/dec & ccu/ccd should match. y-values other than (0,0) may vary.

2. The given graph is g', state several facts about g and g''





g(x) dec on (0,2) $g'<0$
inc on (2,3) $g'>0$

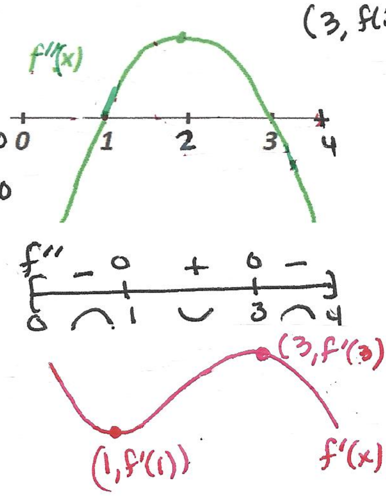
g(2) rel min b/c g' changes ⊖ to ⊕.

g(x) ccd on (0,1) $g''<0$
ccu on (1,3) $g''>0$

(1, g(1)) is point of inflection b/c g'' changes ⊖ to ⊕

3. The given graph is f'', state several facts about f and f'

(1, f(1)) is Inf Point b/c f'' changes ⊖ to ⊕
(3, f(3)) is Inf Pt. b/c f'' changes ⊕ to ⊖



$f'(x)$ is decreasing on $(-\infty,1)$ $(3,\infty)$

$f'(x)$ is increasing on (1, 3)

f' has a max at $(1, f'(1))$

f' has a min at $(3, f'(3))$

Problems involving the first and second derivative

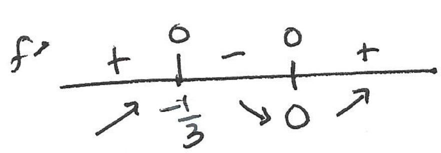
1) Find the critical numbers of each function: $f'(x)=0$ & $f'(x)$ und

a) $f(x)=4x^3+2x^2$

$f'(x)=12x^2+4x$

$f'(x)=4x(3x+1)=0$

$f'(x)=0$ $\underline{x=0}$ $\underline{x=-\frac{1}{3}}$



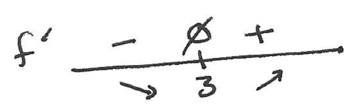
$f(-\frac{1}{3})$ is a Rel MAX b/c f' changes ⊕ to ⊖

$f(0)$ is a Rel Min b/c f' changes ⊖ to ⊕.

b) $f(x)=(x-3)^{\frac{2}{5}}$

$f'(x)=\frac{2}{5}(x-3)^{-3/5}=\frac{2}{5(x-3)^{3/5}}$

$f'(x)=0$ never $f'(x)$ undefined $\underline{@\,x=3}$



$f(3)$ is a Rel MIN b/c f' changes ⊖ to ⊕.

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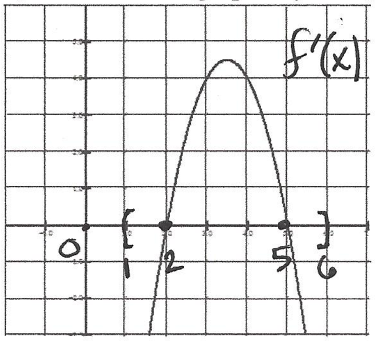
2) Shown is the graph of f' on (1,6). Find the intervals on which f is increasing or decreasing.



f': $-$ 0 $+$ 0 $-$ on [1, 2, 5, 6] : $\searrow$ 2 $\nearrow$ 5 $\searrow$

f is increasing on (2,5) b/c $f'(x) > 0$

f is decreasing on (1,2), (5,6) b/c $f'(x) < 0$.

3) Find the open intervals on which $f(x) = (x^2 - 9)^{\frac{2}{3}}$ is increasing or decreasing. No calculator.

$f'(x) = \frac{2}{3}(x^2-9)^{-1/3}(2x)$

$f'(x) = \dfrac{4x}{3[(x-3)(x+3)]^{1/3}}$

$f'(x) = 0$ @ $x = 0$

$f'(x)$ und @ $x = \pm 3$

f': $-$ $\varnothing$ $+$ 0 $-$ $\varnothing$ $+$: $\searrow$ -3 $\nearrow$ 0 $\searrow$ 3 $\nearrow$

Related Poly
$+(x)(x-3)(x+3)$

f is increasing on $(-3,0)$ $(3,\infty)$ b/c $f' > 0$.

f is decreasing on $(-\infty,-3)$ $(0,3)$ b/c $f' < 0$.

4) The derivative of a function f is given as $f'(x) = \cos(x^2)$. Use a calculator to find the values of x on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ such that f is increasing.

$f'(x) = \cos(x^2) \rightarrow Y_1$

on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Find x-values for which $f'(x) > 0$

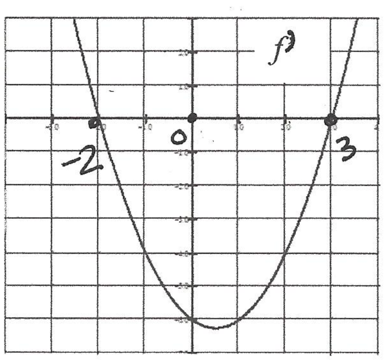
RADIAN MODE!

$f'(x) = 0$ @ $x = \pm 1.2533141$

f': $-$ 0 $+$ 0 $-$ on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$: $\searrow$ c_1 $\nearrow$ c_2 $\searrow$

f(x) is increasing on $(-1.253, 1.253)$ b/c $f' > 0$.

5) The graph of f', the derivative of a function f is shown. Find the relative extrema of f. Justify your answer.



$f'(x) = 0$ @ $x = -2, 3$

$f'(x)$ changes sign $+$ to $-$ @ $x = -2$

$\therefore$ $f(-2)$ is a relative max.

$f'(x)$ changes sign $-$ to $+$ @ $x = 3$

$\therefore$ $f(3)$ is a relative min.

f': $+$ 0 $-$ 0 $+$: $\nearrow$ -2 $\searrow$ 3 $\nearrow$

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6) Find the <u>relative extrema</u> of $f(x)=\frac{x^3}{3}-x^2-3x$. Use the <u>2nd Derivative Test</u>. No calculator.

$f'(x)=x^2-2x-3$

$f'(x)=(x-3)(x+1)=0$

$x=3,-1$

CRITICAL POINTS

$x=3,-1$

$f''(x)=2x-2$

$f''(x)=2(x-1)$

$f''(3)=2(3-1)=4>0$ $\therefore$ f(x) is ccup @ x=3, f(3) is a Rel. MIN.

$f''(-1)=2(-1-1)=-4<0$ $\therefore$ f(x) is ccdown @ x=-1, f(-1) is a Rel. MAX.

7) Find the <u>relative extrema</u> of $f(x)=(x^2-1)^{\frac{2}{3}}$. Use the 1st Derivative Test. No Calculator.

$f'(x)=\frac{2}{3}(x^2-1)^{-1/3}\cdot 2x$

$f'(x)=\frac{4x}{3((x-1)(x+1))^{1/3}}$

C.P. $f'(x)=0$ @ $x=0$ und @ $x=\pm 1$

f': $-$ $\varnothing$ $+$ 0 $-$ $\varnothing$ $+$; $\searrow -1 \nearrow 0 \searrow 1 \nearrow$

f' changes sign ⊖ to ⊕ @ x=-1, 1 $\therefore$ f(1) & f(-1) are Rel MIN.

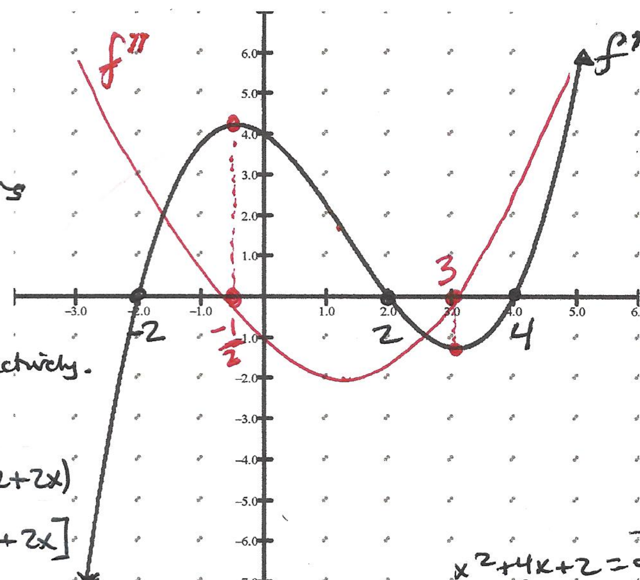
f' changes sign ⊕ to ⊖ @ x=0 $\therefore$ f(0) is a Rel max.

8) The graph of f', the derivative of a function f is shown. Find where the function <u>f is concave up</u>, where it is concave down and where it has points of inflection.

f(x) is concave up when f'(x) has a positive slope or when f'(x) is increasing or when $f''(x)>0$ $\therefore$ on $(-\infty,-\frac{1}{2})$ $(3,\infty)$

f(x) ccdown (f' dec, $f''<0$) on $(-\frac{1}{2},3)$

$\therefore$ Inf. Pts @ $(-\frac{1}{2},f(-\frac{1}{2}))$ & $(3,f(3))$ b/c f'' sign changes. ⊕ to ⊖ & ⊖ to ⊕ respectively.



9) Using a calculator, find the values of x at which the graph of $y=x^2e^x$ <u>changes concavity</u>.

$\frac{dy}{dx}=2xe^x+x^2e^x$

$\frac{dy}{dx}=e^x(2x+x^2)$

$\frac{d^2y}{dx^2}=e^x(2x+x^2)+e^x(2+2x)$

$\frac{d^2y}{dx^2}=e^x[2x+x^2+2+2x]$

$\frac{d^2y}{dx^2}=e^x[x^2+4x+2]=0$

Inf Pt @ $x=-2\pm\sqrt{2}$ where f'' changes signs.

$e^x=0$ never & $x^2+4x+2=0$

$(x+2)^2-2=0$

$(x+2)^2=2$

$x=-2\pm\sqrt{2}$

10) Find the <u>points of inflection</u> of the following functions and determine where the function is <u>concave up</u> and where it is <u>concave down</u>. No calculator.

a) $f(x)=x^3-6x^2+12x-8$

$f'(x)=3x^2-12x+12$

$3(x^2-4x+4)$

$f''(x)=3(2x-4)$

$=6(x-2)=0$

$f''(x)=0$ @ $x=2$

f'': $-$ 0 $+$; $\cap$ 2 $\cup$

f(x) has inflection pt @ (2, 0) b/c f''(x) changes sign from ⊖ to ⊕ @ x=2.

b) $f(x)=(x-1)^{\frac{2}{3}}$

$f'(x)=\frac{2}{3}(x-1)^{-1/3}=\frac{2}{3(x-1)^{1/3}}$

$f''(x)=-\frac{2}{9}(x-1)^{-4/3}=\frac{-2}{9(x-1)^{4/3}}$

denom always >0

$f''(x)=0$ never

$f''(x)$ undefined @ $x=1$

f'': $-$ $\varnothing$ $-$; $\frown$ +1 $\frown$

f(x) has no inflection pts b/c f''(x) never changes signs.