EAGEN KEY

2.6 Continuity and Differentiability—Student Notes HH6ed

Definition: A function $f(x)$ is **continuous at a number *a*** if

$$\lim_{x \to a} f(x) = f(a)$$

This definition implicitly requires three things to be continuous at $x = a$:

Recall from Unit 1:

1. $f(a)$ exists (that is, a is in the domain of $f(x)$)
2. $\lim_{x \to a} f(x)$ exists (so $f(x)$ must be defined on an open interval that contains a)
3. $\lim_{x \to a} f(x) = f(a)$

There are 3 types of Discontinuity:

1. Removable Discontinuity: A limit exists, but there is a hole at the value.
2. Non-removable (or Jump) Discontinuity: A limit does not exist at the value.
3. Infinite Discontinuity: There is a vertical asymptote at the value. The limit from the left and right is ∞ or $-\infty$

Examples:

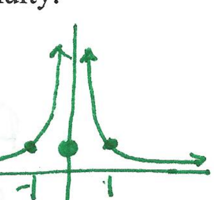
1. Where are each of the following functions discontinuous? State the type of discontinuity.

a. $f(x) = \dfrac{x^2 - x - 2}{x - 2} = \dfrac{(x-2)(x+1)}{(x-2)}$

f(x) is discontinuous at x = 2 where there is a hole (2,3)

∴ REMOVABLE DISCONTINUITY

b. $f(x) = \begin{cases} \dfrac{1}{x^2}, & \text{if } x \neq 0 \\ 1, & \text{if } x = 0 \end{cases}$

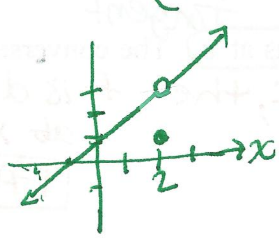


f(x) is discontinuous at x = 0. where there is a vertical asymptote.

∴ INFINITE DISCONTINUITY.

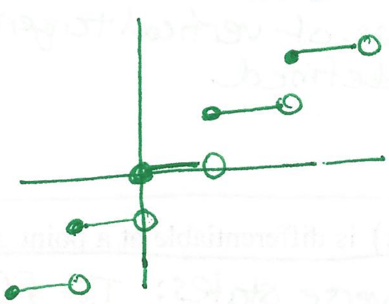
c. $f(x) = \begin{cases} \dfrac{x^2 - x - 2}{x - 2}, & \text{if } x \neq 2 \\ 1, & \text{if } x = 2 \end{cases}$

$f(x) = \begin{cases} x+1, & x \neq 2 \\ 1, & x = 2 \end{cases}$



REMOVABLE DISCONTINUITY

d. $f(x) = [\![x]\!]$ step function has jump discontinuity



Definition A function $f(x)$ is **differentiable at *a*** if $f'(a)$ exists. It is **differentiable on an open interval** (a, b) [or (a,∞) or $(-\infty,a)$ or $(-\infty,\infty)$] if it is differentiable at every number in the interval.

Defn of Derivative FUNCTION $$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

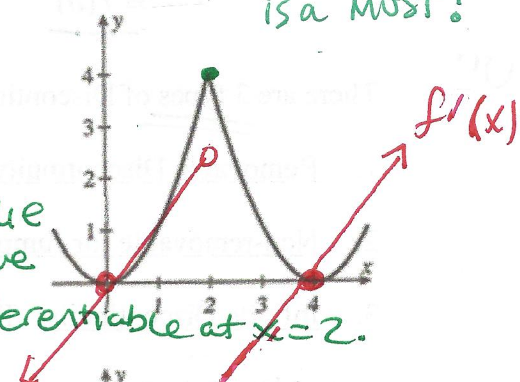
WHEN does a LIMIT EXIST? When LHL = RHL! So for Differentiability LH slope = RH slope is a MUST!

There are 3 common ways for a function to fail to be differentiable at a point

a. The graph has a **sharp point or cusp**.

DRAW DERIVATIVE GRAPHS

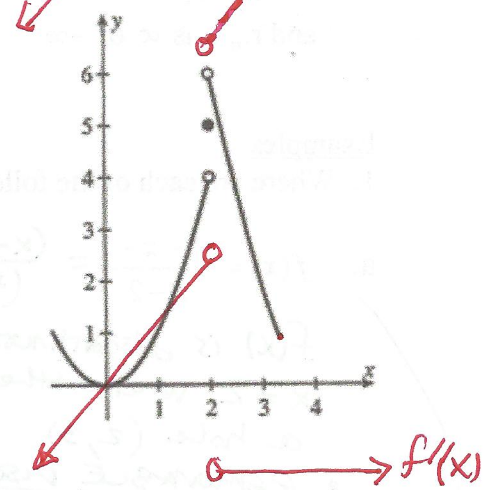
Example: $f(x)=\begin{cases} x^2 \text{ if } x\le 2 \\ (x-2)^2 \text{ if } x>2 \end{cases}$



at x=2 LHL for slope is large positive value
RHL for slope is large negative value
LHL ≠ RHL so f(x) is NOT differentiable at x=2.

b. The function is **discontinuous.** (break, hole or asymptote)

Example: $f(x)=\begin{cases} x^2 \text{ if } x<2 \\ 5 \text{ if } x=2 \\ 10-x^2 \text{ if } x>2 \end{cases}$

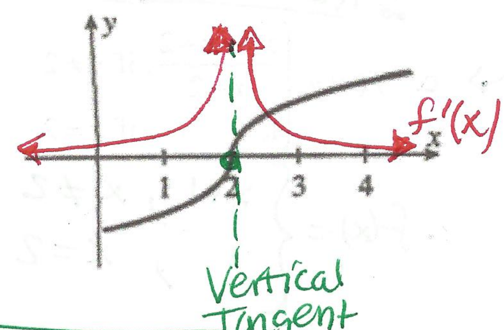


Functions MUST be continuous in order to even consider differentiability.

c. The graph has a **vertical tangent line**.

Example: $f(x)=\sqrt[3]{x-2}$

Slope of vertical tangent is undefined.

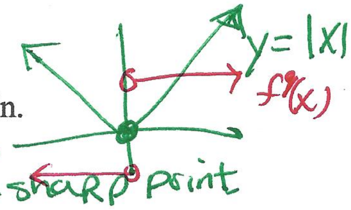


TRUE **Theorem:** If $f(x)$ is differentiable at a point $x=c$, then $f(x)$ is continuous at c. The converse is **false**. The converse states: If f(x) is continuous at c, then f is differentiable at c. FALSE

* differentiability ⇒ continuity ⇒ limit

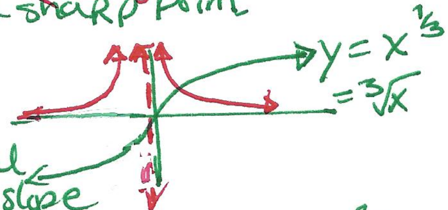
Examples:

2. Is the absolute value function differentiable at $x = 0$? Explain.



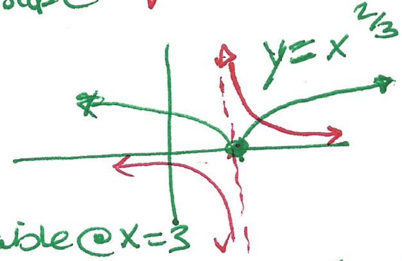
$f(x) = |x|$ is not differentiable @ $x = 0$ b/c $f'(0^-) = -1 \neq f'(0^+) = +1$ b/c there is a sharp point

3. Is $f(x) = x^{\frac{1}{3}}$ differentiable at $x = 0$? Explain.



$f(x)$ is not differentiable at $x = 0$ b/c $f'(0)$ is undefined since there is a vertical tangent with undefined slope

4. Is $f(x) = (x-1)^{\frac{2}{3}}$ differentiable at $x = 1$? Explain.

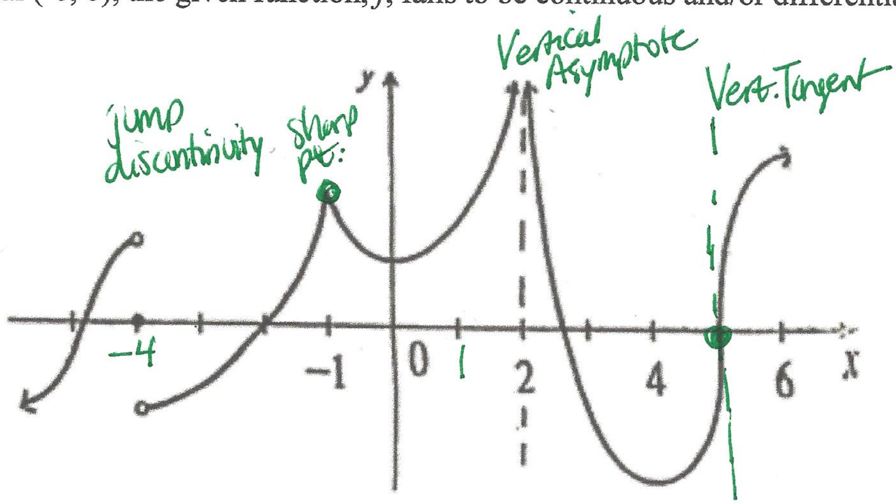


$f(x)$ is not differentiable at $x = 1$ b/c $f'(1^-) \neq f'(1^+)$ b/c there is a sharp point.

5. Is $f(x) = \dfrac{x^2 - 5x + 6}{x - 3}$ differentiable at $x = 3$?

$f(x) = \dfrac{(x-2)\cancel{(x-3)}}{\cancel{(x-3)}}$ $f(x)$ is not differentiable @ $x = 3$ b/c $f(x)$ is discontinuous; there is a hole @ $(3, 1)$.

6. Refer to the figure at the right. Complete the following table indicating at which values on the open interval (-6, 6), the given function, *f*, fails to be continuous and/or differentiable.



Domain value	Continuous? (yes or no)	If no, why?	Differentiable? (yes or no)	If no, why
1. $x = -4$	NO	jump discont. $\lim_{x\to -4^-} f \neq \lim_{x\to -4^+} f(x)$	NO	jump discontinuity
2. $x = -1$	YES	✓	NO	sharp point LHL ≠ RHL
3. $x = 2$	NO	Infinite Discont. Vertical Asymp. $x = 2$	NO	infinite discontinuity
4. $x = 5$	YES		NO	vertical tangent ∴ slope is undefined.