KEY EAGEN

An Alternative Notation for the Derivative and Interpreting its Meaning

Given that $y = f(x)$, the derivative can be written as $f'(x)$ or $\frac{dy}{dx}$.

The second notation was introduced by Wilhelm Gottfried Leibniz, a German mathematician. The letter d stands for "small difference in . . ." so literally the notation $\frac{dy}{dx}$ can be thought of as

$$\frac{\text{Small difference in } y\text{-values}}{\text{Small difference in } x\text{-values}}$$

We say "the derivative of y with respect to x."

LIMIT DEFINITION!

$$\boxed{\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}}$$

Example 1: Use the definition of derivative to find a formula for $\frac{dy}{dx}$ algebraically given $f(x) = x^2 - x$.

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{[(x+h)^2 - (x+h)] - [x^2 - x]}{h}$$

$$= \lim_{h \to 0} \frac{(\cancel{x^2} + 2xh + h^2 - \cancel{x} - h) - (\cancel{x^2} - \cancel{x})}{h}$$

$$= \lim_{h \to 0} \frac{2xh + h^2 - h}{h}$$

$$= \lim_{h \to 0} (2x + h - 1)$$

$$\frac{dy}{dx} = 2x - 1$$

$f(x) = x^2 - x$

$f'(x) = 2x - 1$

If we want to indicate that you should find the derivative at $x = 2$, you write $f'(2)$ or $\left.\frac{dy}{dx}\right|_{x=2} = 2(2) - 1 = 3$

Example 2: Suppose $s = f(t)$ gives the distance, in meters, of a body from a fixed point as a function of time t, in seconds.

$s(t)$ describes position. $s'(t)$ describes velocity.
$s''(t)$ describes acceleration.

a. Describe the following in real-world terms: $\left.\frac{ds}{dt}\right|_{t=2}$

at time $t = 2$ seconds, the body is moving at a velocity of $\frac{ds}{dt}$ $\frac{\text{meters}}{\text{sec.}}$

b. What are the units associated with this quantity? m/sec.

c. What is the common term for $\frac{ds}{dt}$? instantaneous velocity.

d. What is the real-world meaning of $f'(2) = 10$? Use units in your answer.

Interpret within context!!!

The position of the object from the fixed point is increasing 10 m/sec at time $t = 2$ seconds.

Example 3: The cost, C, in dollars, of building a house A ft^2 in area is given by the function $C = f(A)$.

a. What is the real world meaning of $f(2000) = 195{,}000$? Use units in your answer.

To build a 2000 ft^2 house will cost \$195,000.

b. What is in the independent variable? Dependent variable?

A - area of house (in square feet) C - cost of house (in \$.)

c. What is the sign of $f'(A)$? Why?

$f'(A) > 0$ positive because as the area increases the cost increases.

d. Rewrite $f'(A)$ in Leibniz's notation.

$$f'(A) = \frac{dC}{dA}$$

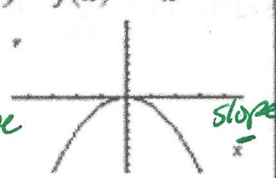
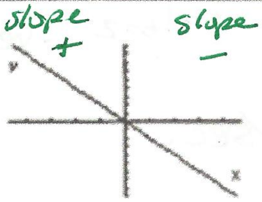
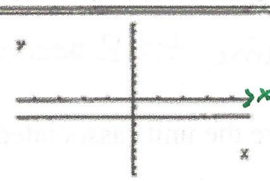
e. What are the units of $f'(2000)$?

dollars per square ft.: $\frac{\$}{ft^2}$

f. What is the real world meaning of $f'(2000) = 150$?

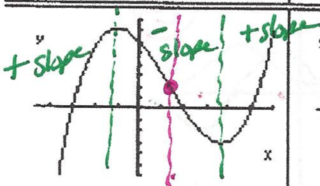
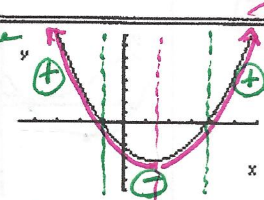
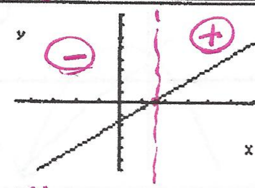
For an area of 2000 square feet, the cost to build is increasing at a rate of \$150/sq ft.

Analyze the graph of $y = -x^2$, using the first and second derivative graphs.

Graph of f(x)	Graph of f '(x)	The value of the slope of tangent line	Graph of f ''(x)	Observations
$y = f(x) = -x^2$ slope + slope -  • for $x < 0$ f is increasing • for $x > 0$ f is decreasing • f is always concave down	slope + slope -  • for $x < 0$ $f'(x)$ is positive. • for $x > 0$ $f'(x)$ is negative. • $f'(x)$ is always decreasing the slope of the slope graph is negative	$f'(-2) = 4$ $f'(-1) = 2$ $f'(0) = 0$ $f'(1) = -2$ $f'(2) = -4$ $f'(x)$ is always decreasing	 $f''(x) < 0$ for all x.	On the same interval: • $f(x)$ is concave down • $f'(x)$ is decreasing • $f''(x) < 0$

$f(x) = -x^2$ $f'(x) = -2x$ $f''(x) = -2$

Observe the relationship between the graphs of $f(x)$, $f\,'(x)$ and $f\,''(x)$.

Graph of f(x)	***Graph of f'(x)***	***Graph of f''(x)***	***Observations***
 *f(x) changes concavity* ccd ccu − +	 *f '(x) decreases then increases*	 *f ''(x) is negative then positive*	• *Where f ''(x) changes from negative to positive, f(x) changes concavity.* • *Where f ''(x) = 0, f(x) changes concavity.* • *Where f '(x) changes from decreasing to increasing, f ''(x) = 0.*

Conclusions:

The First Derivative:

- If $f'(x)>0$ on an interval, then $f(x)$ is <u>increasing</u> over that interval.
  positive slope function
- If $f'(x)<0$ on an interval, then $f(x)$ is <u>decreasing</u> over that interval.
  negative slope function
- If $f'(x)=0$ on an interval, then $f(x)$ has a <u>horizontal tangent</u> at x which is

  either a <u>turning point</u> when <u>$f'(x)$</u> changes sign $f'(x)$ → + to −, → − to +

  or a <u>terrace point</u> when <u>$f'(x)$</u> does not change sign. $f'(x)$ → + to +, → − to −

The Second Derivative:

- If $f''(x)>0$ on an interval, then $f'(x)$ is <u>increasing</u> and
  positive

  $f(x)$ is <u>concave up</u> over that interval.
- If $f''(x)<0$ on an interval, then $f'(x)$ is <u>decreasing</u> and
  negative

  $f(x)$ is <u>concave down</u> over that interval.
- If $f''(x)=0$ on an interval, then $f(x)$ sometimes has a <u>point of inflection</u> (change in concavity) at

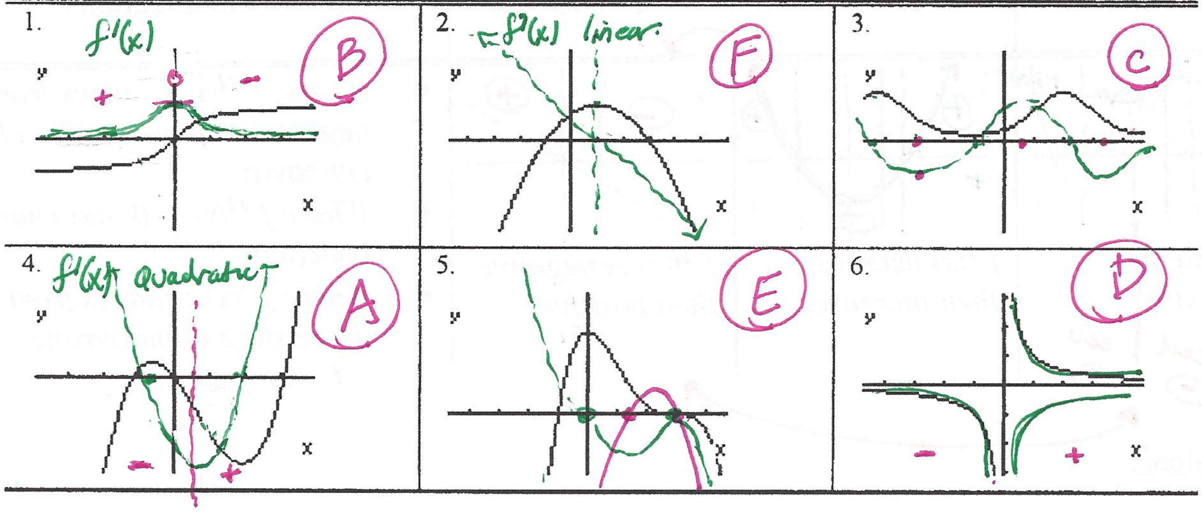
  that value of x, but only if <u>$f''(x)$</u> changes sign.

  $f''(x)$ → + to −, → − to +

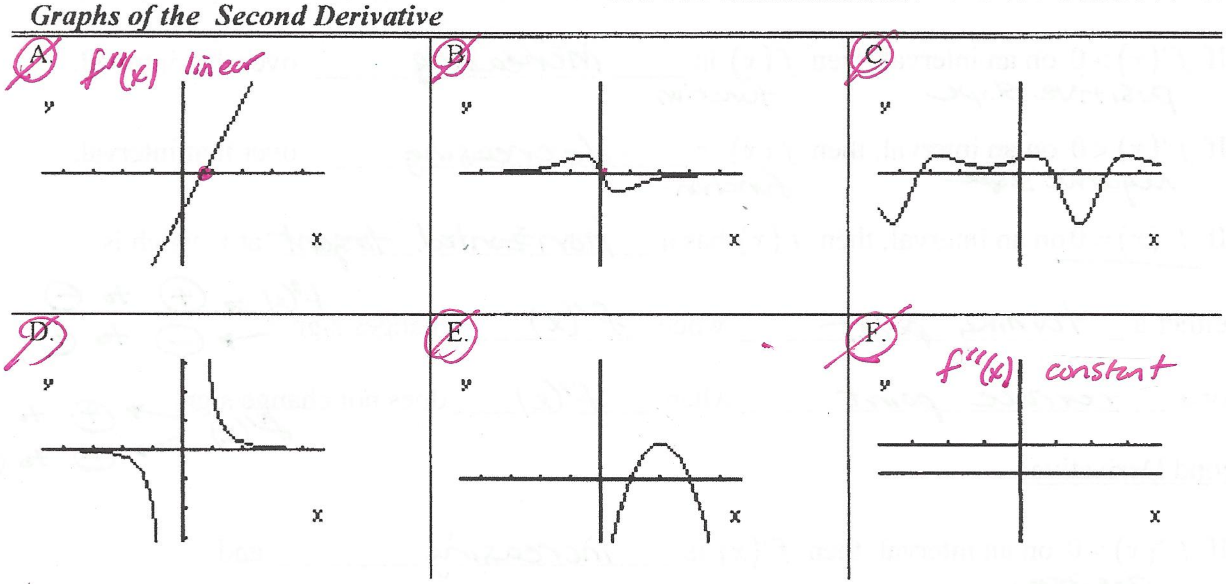
DRAW 1st DERIVATIVE

Match the graph of each function below 1 - 6, with the graph of its second derivative A - F.

Graphs of the Function



Graphs of the Second Derivative



Determine which of the functions graphed below is

a) increasing at an increasing rate.

b) increasing at a decreasing rate.

c) decreasing at an increasing rate or

d) decreasing at a decreasing rate.

Explain why you have chosen each.

Graphs of Increasing and Decreasing functions