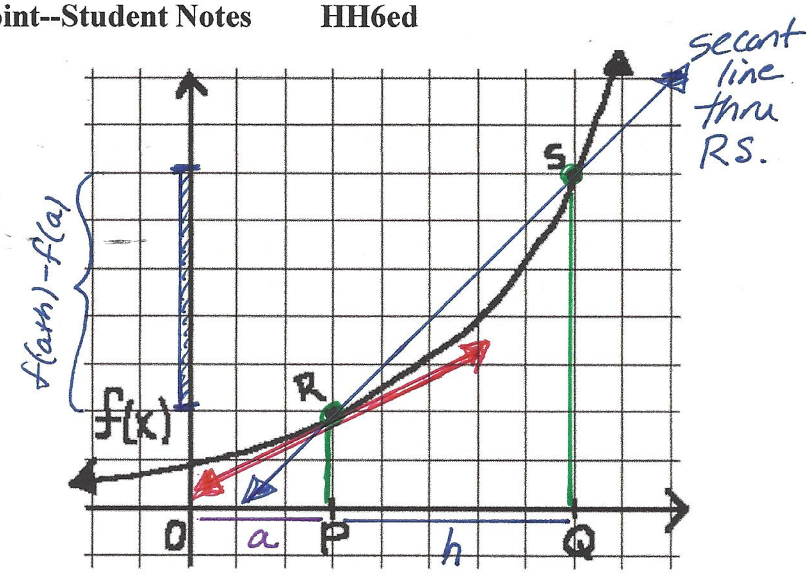
EAGEN ANSWER KEY

2.2 The Derivative at a Point--Student Notes HH6ed



Refer to the graph at the right of some arbitrary function *f*.

1. Let a represent the distance from the origin O to the point P. Label it on the graph. Identify coordinate P

$P(\underline{a}, \underline{0})$

2. Let h represent the distance from point P to point Q. Label it on the graph. Outline it in blue Identify coordinate Q.

$Q(\underline{a+h}, \underline{0})$

3-4. Outline segments $\overline{RP}$ *and* $\overline{SQ}$ in green.

Write the algebraic expressions for the lengths of $\overline{RP}$ *and* $\overline{SQ}$ and identify coordinate R and S.

$RP = \underline{f(a)}$ $SQ = \underline{f(a+h)}$ $R(\underline{a}, \underline{f(a)})$ $S(\underline{a+h}, \underline{f(a+h)})$

5. On the figure draw and label the segment whose length is $f(a+h)-f(a)$ in blue. vertical distance between R & S.

intersects curve twice.

6. Draw the secant line RS in blue. Write an algebraic expression for its slope. Simplify completely. ✓

$$\frac{\Delta y}{\Delta x} = \frac{f(a+h)-f(a)}{(a+h)-a} = \frac{f(a+h)-f(a)}{h}$$

7. Suppose you were to take the limit of the slope expression you just wrote as h gets infinitely small. What would this limit represent geometrically?

$$\lim_{h\to 0}\left(\frac{f(a+h)-f(a)}{h}\right)$$

The SLOPE of the TANGENT line at R.

So point S would move closer & closer & closer to point R & the secant line would become a TANGENT line at pt. R.

8. Sketch the tangent line to the function f at the point R in red.

9. Write an algebraic expression for the slope of this line (Hint: Recall the relationship between average velocity and instantaneous velocity.)

$$f'(a) = \lim_{h\to 0}\left(\frac{f(a+h)-f(a)}{h}\right)$$

10. What notation do we use for this quantity?

$f'(a)$ is the instantaneous velocity or instantaneous

11. What special name do we reserve for this quantity? rate of change at the point $(a, f(a))$

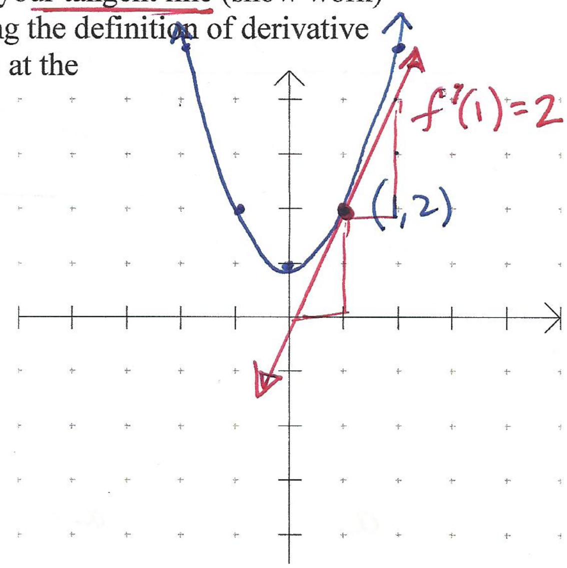
Conclusion: The instantaneous rate of change or the derivative is $f'(a) = \lim\limits_{h\to 0}\dfrac{f(a+h)-f(a)}{h}$

The derivative at a point $(a, f(a))$ is $f'(a)$

Conclusion: The instantaneous rate of change or the derivative is $f'(a) = \lim\limits_{h \to 0} \dfrac{f(a+h) - f(a)}{h}$

Practice: For each function, make a sketch of the curve and use your straight edge to draw the tangent line to the curve at the give point.

a. Estimate the slope of the curve at the point using your tangent line (show work)
b. Find the actual slope of the curve at the point using the definition of derivative
c. Write the equation of the tangent line to the curve at the point.



12. $f(x) = x^2 + 1$ at $x = 1$

a. $f'(1) = 2$ based on tangent line

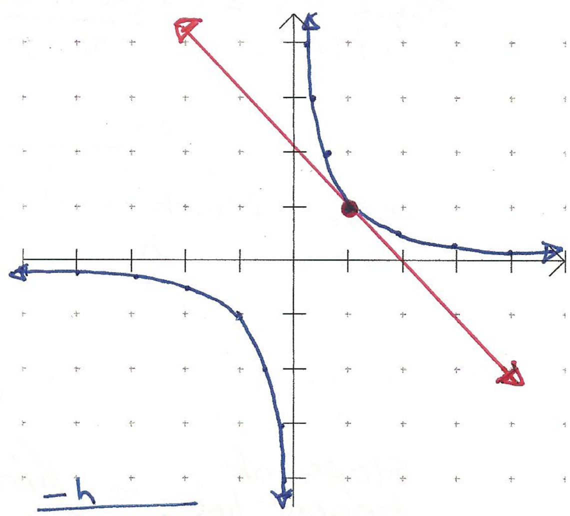
b. $f'(1) = \lim\limits_{h \to 0} \dfrac{f(1+h) - f(1)}{h}$

$= \lim\limits_{h \to 0} \dfrac{[(1+h)^2 + 1] - [1^2 + 1]}{h}$

$= \lim\limits_{h \to 0} \dfrac{(1 + 2h + h^2 + 1) - (2)}{h}$

$= \lim\limits_{h \to 0} \dfrac{2h + h^2}{h} = \lim\limits_{h \to 0} (2 + h) = 2$

c. $f'(1) = 2$ & $(1, 2)$ $\boxed{y = 2(x-1) + 2}$ tangent line



13. $f(x) = \dfrac{1}{x}$ at $x = 1$

a. $f'(1) = -1$ based on tangent line

b. $f'(1) = \lim\limits_{h \to 0} \dfrac{f(1+h) - f(1)}{h}$

LCD

$= \lim\limits_{h \to 0} \dfrac{\left(\frac{1}{1+h} - \frac{1}{1}\right)}{(h)} \cdot \dfrac{(1+h)}{(1+h)}$

$= \lim\limits_{h \to 0} \dfrac{1 - (1+h)}{(h)(1+h)} = \lim\limits_{h \to 0} \dfrac{-h}{h(1+h)}$

$= \lim\limits_{h \to 0} \dfrac{-1}{(1+h)} = -1$

c. $f'(1) = -1$ & $(1, 1)$ $\boxed{y = -1(x-1) + 1}$ tangent line

EAGEN KEY

14. Find the derivative of $f(x) = 5x^2$ at $x = 10$ using the definition of derivative.

$$f'(10) = \lim_{h \to 0} \frac{f(10+h) - f(10)}{h}$$

$$= \lim_{h \to 0} \frac{5(10+h)^2 - 5(10)^2}{h} = \lim_{h \to 0} \frac{5(\cancel{100}+20h+h^2) - 5(\cancel{100})}{h}$$

$$= \lim_{h \to 0} \left(\frac{100h + 5h^2}{h}\right) = \lim_{h \to 0} 100 + 5h = 100$$

$f'(10) = 100$

15. Find the equation of the line tangent to the function $f(x) = x^3$ at $x = -2$ using the definition of derivative.

$$f'(-2) = \lim_{h \to 0} \frac{f(-2+h) - f(-2)}{h} = \lim_{h \to 0} \frac{(-2+h)^3 - (-2)^3}{h}$$

$$= \lim_{h \to 0} \frac{(\cancel{-8}+12h-6h^2+h^3) - (\cancel{-8})}{h} = \lim_{h \to 0} \frac{12h-6h^2+h^3}{h}$$

$$= \lim_{h \to 0} (12 - 6h + h^2) = 12$$

$f'(-2) = 12 \quad f(-2) = -8$

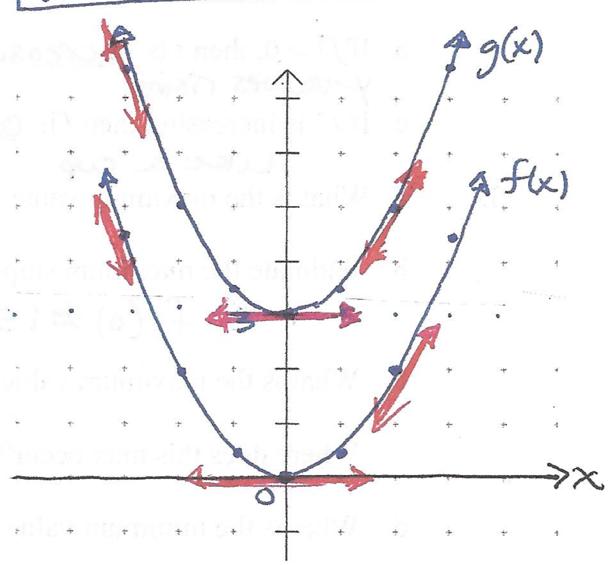
$\boxed{y = 12(x+2) - 8}$ tangent line

Cube:

$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + 1b^3$

$(-2+h)^3 = -8 + 12h - 6h^2 + h^3$

16. a. Sketch the graphs of the functions $f(x) = \frac{1}{2}x^2$ *and* $g(x) = f(x) + 3$ on the same set of axes.



The slopes of the tangent lines at $x = 0, 2$ & any c will be the same for $f(x)$ & $g(x)$

b. What can you say about the slopes of the tangent lines to the two graphs at the point $x = 0$? $x = 2$? $x =$ any c?

they are the same.

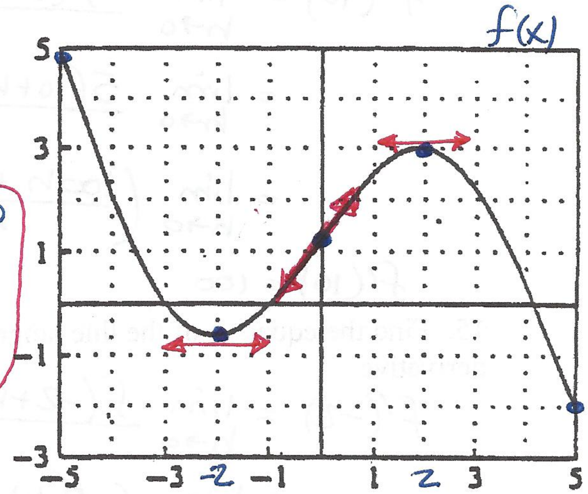
c. Explain why adding a constant value, c, to any function does not change the value of the slope of its graph at any point.

The shape of the graph does not change by a vertical translation so the slopes will be the same.

EAGEN KEY $f(x)$ = y-value
$f'(x)$ = slope value

Review of Terminology: Refer to $f(x)$ with domain $[-5,5]$ to answer the following questions.

1. Specify the intervals on which



Graph of f

a. f is positive $f(x)>0$
$(-5,-3)$ $(-1,4)$

f. f is negative $f(x)<0$
$(-3,-1)$ $(4,5)$

NOTICE

b. f' is positive $f'(x)>0$
$(-2,2)$

g. f' is negative $f'(x)<0$
$(-5,-2)$ $(2,5)$

c. f is increasing
$(-2,2)$

h. f is decreasing
$(-5,-2)(2,5)$

d. f' is increasing
$(-5,0)$

i. f' is decreasing
$(0,5)$

e. f is concave up
$(-5,0)$

j. f is concave down
$(0,5)$

2. Complete the following statements using the problem above:

a. If $f' > 0$, then f is increasing
y-values rising

b. If $f' < 0$, then f is decreasing
y-values falling

c. If f' is increasing then f is concave up
Like a cup

d. If f' is decreasing then f is concave down
Like a frown

3. a. What is the maximum value of f? 5 Where does it occur? $x=-5$

b. Estimate the maximum slope of f'. Explain. At what x-value does this occur?
$f'(0) \approx 1.5$ or 2 at $x=0$

c. What is the maximum value of f on the interval [0, 4]? 3

Where does this max occur? $x=2$ What is the value of f' at this point? $f'(2)=0$

d. What is the minimum value of f on the interval [-5, 0]? -0.7

Where does this min occur? $x=-2$ What is the value of f' at this point? $f'(-2)=0$

4. In each of the given pairs of expressions, determine which is larger. Explain.

a. $f(2)$ or $f(3)$
$f(2)=3$ $f(3)=2.2$

b. $f'(2)$ or $f'(3)$
$f'(2)=0$ $f'(3)=-1$

c. $f(1)-f(0)$ or $f(2)-f(1)$
$f(1)-f(0)=2.5-1=1.5$
$f(2)-f(1)=3-2.5=0.5$

d. $\dfrac{f(1)-f(0)}{1-0}$ or $\dfrac{f(2)-f(0)}{2-0}$
$=\dfrac{1.5}{1}$ $=\dfrac{0.5}{2}$
$=\dfrac{3}{2}$ $=\dfrac{1}{4}$