ANSWER KEY EAGEN

2.1 How Do We Measure Speed?—Student Notes HH6ed

Part I: Using a table of values for a position function

The table below represents the position of an object as a function of time. Use the table to answer the questions that follow.

$(t, s(t))$
(seconds, meters)

Time (sec) t	Position (m) $s(t)$
2.8	7.84
2.9	8.41
3.0	9.00
3.1	9.61
3.2	10.24
3.3	10.89

#5 #7a) #7b)

1. What is the object's position at time $t = 3$ sec? at time $t = 3.3$ sec?

$s(3) = 9$ meters. $s(3.3) = 10.89$ m.

2. What is the *total change* in the object's position over the time interval from 3 to 3.3 sec?

$s(3.3) - s(3) = 1.89$ m

3. Find the *average rate of change* in the object's position over the time interval from 3. to 3.3 sec. Show your work. Include units.

$\frac{s(3.3)-s(3)}{3.3-3} = \frac{1.89\text{ m}}{0.3\text{ sec}} = 6.3\ \frac{\text{m}}{\text{sec}}$

4. By what familiar name do we refer to *average rate of change in position?*

AROC = average velocity = slope of secant line.

5. Estimate the *instantaneous rate of change* in the object's position at time $t = 3$ sec. Show work. Include units.

$\frac{s(3.1)-s(2.9)}{3.1-2.9} = \frac{9.61-8.41}{3.1-2.9} = \frac{1.2}{.2} = 6\ \frac{\text{m}}{\text{sec}}$

6. By what familiar name do we refer to *instantaneous rate of change of position*?

IROC = instantaneous velocity = slope of tangent line.

7. Find two other reasonable estimates for the object's velocity at time $t = 3$ sec. Show work.

7a) $\frac{s(3)-s(2.9)}{3-2.9} = \frac{9-8.41}{.1} = 5.9$ m/sec

OR

7b) $\frac{s(3.1)-s(3)}{3.1-3} = \frac{9.61-9}{.1} = 6.1$ m/sec

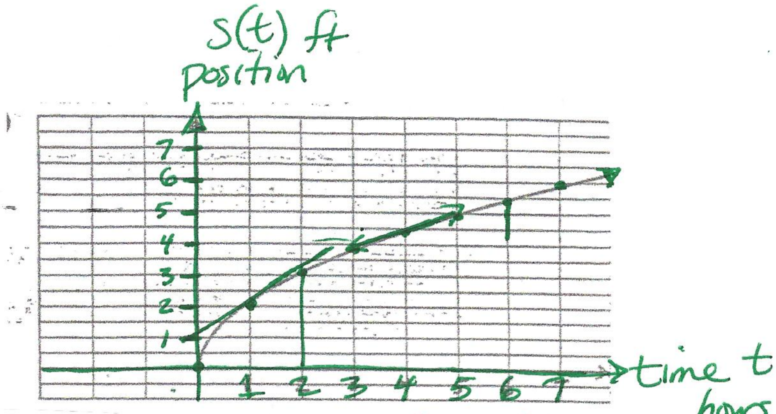
8. Of your three estimates for velocity at $t = 3$ sec, which one do you prefer? Why?

* Best estimate is shown in #5.
Use values surrounding $(3, s(3))$

LH Est.	Best Est	R.H. Est.
5.9 m/sec	6 m/sec	6.1 m/sec

Part 2: Using a graph of a position function

The graph shown represents the object's position, in miles, as a function of time, in hours since noon. (The vertical scaling is intentionally omitted.)



1. Does the object cover a greater distance over the two-hour time interval beginning at noon or over the two-hour time interval beginning at 4:00 p.m.? Explain.

✓ $t \in [0, 2] \rightarrow$ steeper, greater distance since $s(2) - s(0) > s(6) - s(4)$

× $t \in [4, 6] \rightarrow$ not as steep, shorter distance.

2. Does the object have a greater average velocity over the two-hour time interval beginning at noon or over the two-hour time interval beginning at 4:00 p.m.? Explain

✓ $t \in [0, 2]$ AROC $\frac{s(2) - s(0)}{2} > \frac{s(6) - s(4)}{2}$

larger slope

3. Is the object traveling faster at 1:00 p.m. or at 4:00 p.m.? Explain.

AROC = Avg Velocity = slope is greater at 1 pm than at 4pm.

✓ Label the scale of the y-axis so that 1 block = ½ mile.

4. Refer back to question #1, but now calculate the distances covered over the two given time intervals. Then decide if your answer to question #1 was correct.

$s(2) - s(0)$ $= 3 - 0 = 3$ miles

$s(6) - s(4)$ $5.5 - 4.5 = 1$ mile

5. Refer back to question #2, and calculate the average velocities over the two time intervals. Draw the secant lines on the graph. Show work (including units) and decide if your answer to #2 was correct.

$\frac{s(2) - s(0)}{2 - 0} = \frac{3\text{ m}}{2\text{ hr}} = 1.5\text{ m/hr}$

$\frac{s(6) - s(4)}{6 - 4} = \frac{1\text{ m}}{2\text{ hr.}} = 0.5\text{ m/hr}$

6. Refer back to question #3 and estimate the instantaneous velocities at the two specified points in time. Draw the tangent lines. Show work (including units) and decide if your answer to #3 was correct.

at $t = 1$pm $\frac{s(2) - s(0)}{2 - 0} = \frac{3 - 0}{2 - 0} = 1.5 \frac{\text{m}}{\text{hr}}$

at $t = 4$pm $\frac{s(5) - s(3)}{5 - 3} = \frac{5 - 4.8}{2} = \frac{.2}{2} = .1 \frac{\text{m}}{\text{hr}}$

Yes my answer to #3 was correct.

b/c $1.5 \frac{\text{m}}{\text{hr}} > .1 \frac{\text{m}}{\text{hr}}$

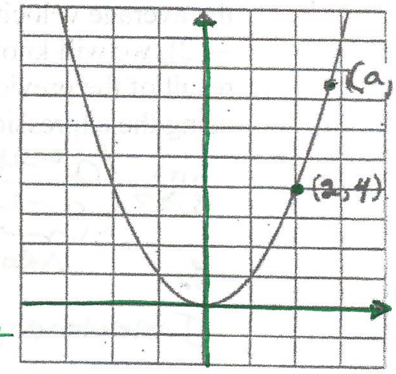
CALCULATOR HOMESCREEN $\boxed{\#}$ STO▸ A: $(A^2-4)/(A-2)$ $\boxed{\text{ENTER}}$
2ND ENTER
2ND LEFT ARROW
2ND DEL (INSERT)

OR

$Y_1=(X^2-4)/(X-2)$
HOMESCREEN
$Y_1(\#)$ ENTER

position ⇒ distance

Part 3: Moving closer to a formal definition of instantaneous velocity



Consider the function $s(t) = t^2$ shown on the graph below. Suppose this function represents the position (meters) of an object at time t (seconds). How can we find the object's instantaneous velocity at a particular point in time, for example at $t = 2$? Is it even possible to do so? On the given function s, the point (2, 4) has been labeled along with a second arbitrary point (a, a^2). Answer the following questions. Many of the answers will be expressions in terms of a.

1. What does the quantity $s(2)$ represent? What is its value?

   s(2) is the distance or position of the object in meters at time t=2 seconds.
   $s(2)=(2)^2=4$ meters.

2. What does the quantity $s(a)$ represent? What is its value?

   s(a) is the position of the object in meters at time t=a seconds.
   $s(a)=(a)^2=a^2$ meters.

3. Write an expression for the total change in the object's position on the time interval $[2, a]$.

   $\Delta s(t) = s(a) - s(2)$
   $= a^2-4$ meters

4. Write an expression for the object's average velocity on the time interval $[2, a]$.

   Avg Vel = AROC = $\frac{s(a)-s(2)}{(a-2)} = \frac{a^2-4}{a-2}$ $\frac{\text{meters}}{\text{sec.}}$

5. We can use the object's average velocity on the interval $[2, a]$ to approximate the object's instantaneous velocity exactly at the time $t = 2$. Of course, as the interval $[2, a]$ gets smaller and smaller (i.e., as the value of a gets closer and closer to 2), the closer the average velocity will approximate the instantaneous velocity. $\left(\frac{a^2-4}{a-2}\right)$ $\left(\frac{4-a^2}{2-a}\right)$

a. Calculate the average velocity of the object on the following time intervals. Show work.

$[2,a]$
as a gets closer & closer to 2
"$a \to 2$"
the AROC gets closer to the IROC.

[2, 2.1]	$\frac{(2.1)^2-4}{2.1-2}=4.1\ \frac{m}{sec}$	[1.9, 2]	$\frac{4-(1.9)^2}{2-1.9}=3.9\ \frac{m}{sec}$
[2, 2.01]	$\frac{(2.01)^2-4}{2.01-2}=4.01\ \frac{m}{sec}$	[1.99, 2]	$\frac{4-(1.99)^2}{2-1.99}=3.99\ \frac{m}{sec}$
[2, 2.001]	$\frac{(2.001)^2-4}{2.001-2}=4.001\ \frac{m}{sec}$	[1.999. 2]	$\frac{4-(1.999)^2}{2-1.999}=3.999\ \frac{m}{sec}$
[2, 2.0001]	$\frac{(2.0001)^2-4}{2.0001-2}=4.0001\ \frac{m}{sec}$	[1.9999, 2]	$\frac{4-(1.9999)^2}{2-1.9999}=3.9999\ \frac{m}{sec}$

from RHS from LHS

NOTICE LHL = RHL

as $a \to 2$ AROC $\to$ IROC

$\lim_{a\to 2} \frac{a^2-4}{a-2} = 4 \frac{m}{sec}$

b. Just how small must we make the interval [2, *a*] in order to get the exact value for the instantaneous velocity at $t = 2$? The answer **is infinitely small!** If we find the limit of the average velocity as the time interval [2, *a*] shrinks to zero (i.e., as $(a - 2) \to 0$ or $a \to 2$), we will know the exact value of the instantaneous velocity at time $t = 2$. Use the result of the previous question to estimate the answer. Then, find the limit algebraically, using the expression for average velocity that you wrote in #4.

$$\lim_{a\to 2}\left(\underbrace{\frac{a^2-4}{a-2}}_{\text{AROC}}\right) = \lim_{a\to 2}\left(\frac{(a-2)(a+2)}{(a-2)}\right) = \lim_{a\to 2}(a+2) = \underbrace{4\ \frac{m}{sec.}}_{\text{IROC}}$$

$\therefore$ Instantaneous Velocity at $t = 2$ sec is $4\ m/sec$.

c. What is the sign of the average velocities on the interval $(-\infty, 0)$? Why?

The velocity values will be negative on $(-\infty, 0)$ because the slope is negative & the distance or position of the object is decreasing.

d. What is the sign of the average velocities on the interval $(0, \infty)$? Why?

The velocity values will be positive on $(0, \infty)$ because the slope is positive & the distance or position of the object is increasing.

So, formally average velocity is the ratio of a change in position (distance) to a change in time. Velocity can be positive, zero or negative, depending on the direction traveled. If two points on the position function $s(t)$ are $(a, s(a))$ and $(b, s(b))$ then

AROC: $$\boxed{\text{Average velocity} = v(t) = \frac{\textit{change in position}}{\textit{change in time}} = \frac{\Delta s}{\Delta t} = \frac{s(b)-s(a)}{b-a}}$$

Speed is the magnitude of velocity and is always positive or zero. Speed = |Velocity|

absolute value represents magnitude.

When you drive around town you calculate your average speed because you are not concerned about the direction you are traveling, only the distance you are traveling.

Instantaneous velocity refers to the velocity at a particular point in time.

IROC: $$\boxed{\text{Instantaneous velocity} = \lim_{b\to a}\frac{s(b)-s(a)}{b-a}}$$

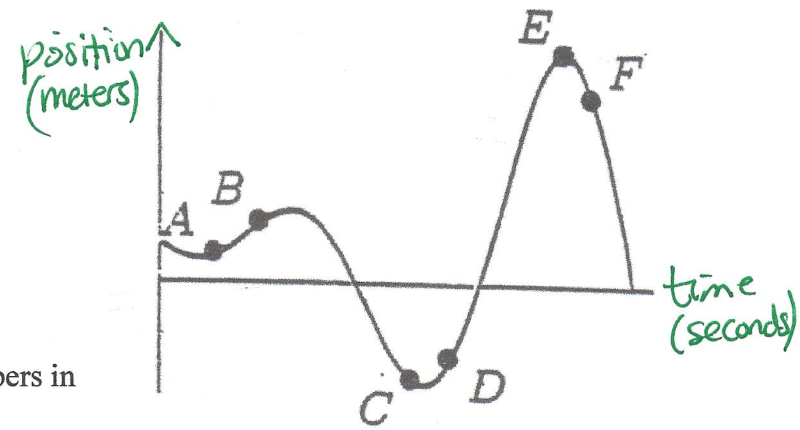
as $b \to a$ the change $(b-a)$ becomes smaller & smaller... $(b-a) \to 0$ infinitely small or almost zero.

It is also the slope of the tangent line to the curve at that point.

Part 4: How Do We Measure Speed? How About Velocity?

1. Match the points labeled on the curve with the given slopes.

Slope	Point
-3	F
-1	C
0	E
½	A
1	B
2	D

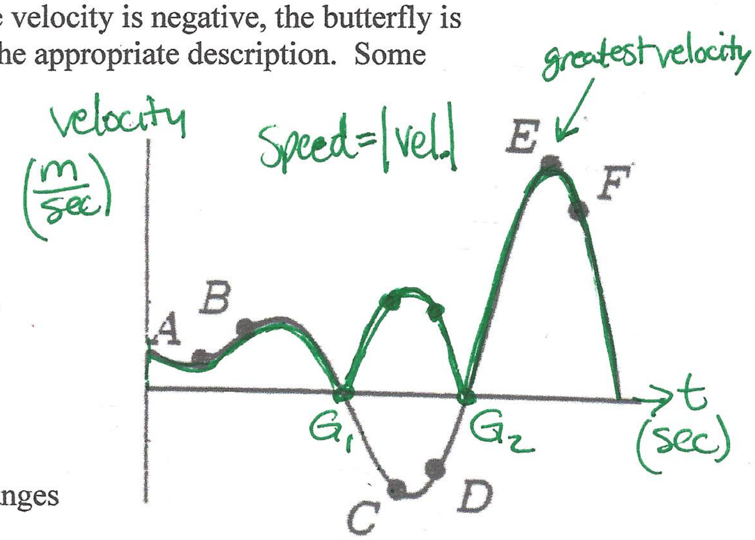


2. For the graph, arrange the following numbers in ascending order.

1 2 3 4 5
− − 0 + +

4 the slope of the graph at A >0
3 the slope of the graph at E =0
1 the slope of the line EF <0
5 the slope of the line AB >0
2 the slope of the graph at C <0

3. Suppose that the graph is of the velocity vs. time of a butterfly in flight. When the velocity is positive, the butterfly is flying upward. When the velocity is negative, the butterfly is flying downward. Match the labeled point(s) with the appropriate description. Some descriptions may fit more than one point.



E the butterfly is flying the fastest SPEED
F the butterfly's velocity is increasing but at a decreasing rate
C,D the butterfly is in a dive toward a tasty flower
A the butterfly is flying the slowest SPEED
C,F the butterfly's velocity is decreasing
D,F the butterfly's speed is decreasing

4. Find a point on the graph where the butterfly changes direction. Label it G. There may be more than one.

X-intercepts
G_1, G_2

Point: A,B,E,F
velocity >0

point: C,D
velocity <0