Six Standard Related Rates problems that you are expected to know how to solve.

1) **The Sliding Ladder Problem**: The top of a fifteen-foot ladder is sliding down a wall while the foot of the ladder is sliding along the floor. If the top of the ladder is sliding down the wall at the rate of 2 feet per second,

a) with what speed is the foot of the ladder sliding along the floor when the foot of the ladder is 5 feet from the wall?

(Answer: $4\sqrt{2}$ **ft/sec)**

b) at what rate is the angle of elevation of the ladder changing then?

(Answer: $-2/5$ **rad/sec)**

2) **The Shadow Problem**: A man 2 meters tall walks at the rate of 2 meters per second toward a streetlight that's 5 meters above the ground. At what rate is the length of his shadow changing?

(Answer: -4/3 meters/sec)

3) **Cones**: A truck is dumping sand into a conical pile a the rate of 60 cubic feet per second and in such a way that the height of the pile is always equal to three times the radius of the base. At what rate is the height of the sand pile changing when the radius of the base of the pile is 20 feet?

(Answer: $3/(20\pi)$ **ft/ sec)**

4) **Circles:** A stone thrown into a pond creates a circular ripple. If the radius of the circle is increasing at the rate of 2 feet per second, at what rate is the area of the circle increasing when the area is 100 square feet?

(Answer: $40\sqrt{\pi}\ \text{ft}^2/\text{sec}$ **)**

5) **Moving Particle**: A particle is sliding down the curve $y = \frac{1}{x}$. When it is at the point (1, 1) on the curve,

$\frac{dy}{dt} = -2$. Find $\frac{dx}{dt}$ at this instant.

(Answer: 2)

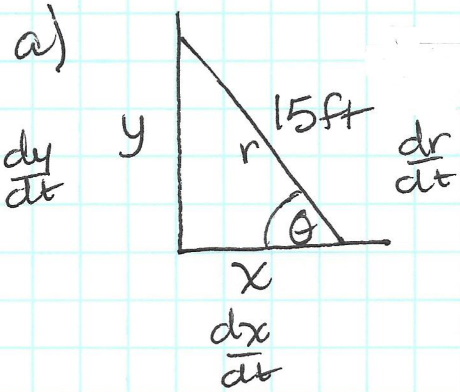
6) **Vehicle Intersection**: Car A passes through an intersection, traveling east, at 10 am and maintains a constant velocity of 60 miles per hour. Car B travels north through the same intersection at 11 am and maintains a constant velocity of 50 miles per hour. How fast is the distance between them changing at noon?

(Answer: about 74.6 miles per hour)

6 RELATED RATES PROBLEMS

① SLIDING LADDER:

a)



$$\frac{dy}{dt} = -2\ \frac{ft}{sec}$$

$$\frac{dx}{dt} = ?\ \frac{ft}{sec}$$

$$\frac{dr}{dt} = 0\ \frac{ft}{sec.}$$

$$x^2 + y^2 = r^2$$

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2r\frac{dr}{dt}$$

$$x\frac{dx}{dt} + y\frac{dy}{dt} = r\frac{dr}{dt}$$

$$x\frac{dx}{dt} = \left(r\frac{dr}{dt} - y\frac{dy}{dt}\right)$$

$$\frac{dx}{dt} = \left(\frac{1}{x}\right)\left(r\frac{dr}{dt} - y\frac{dy}{dt}\right)$$

When $x = 5$ ft

$$y = \sqrt{15^2 - 5^2}$$

$$y = \sqrt{5^2(3^2 - 1^2)}$$

$$y = 5\sqrt{8}$$

$$y = 10\sqrt{2}\ \text{ft.}$$

$$\frac{dx}{dt} = \frac{1}{5}\left(15\cdot(0) - 10\sqrt{2}(-2)\right)$$

$$\frac{dx}{dt} = +4\sqrt{2}\ ft/sec$$

ATQ: The ladder is sliding away from the wall at a rate of $4\sqrt{2}$ ft/sec when $x = 5$ft.

b) $\frac{d\theta}{dt} = ?$

$$\tan\theta = \frac{y}{x} \qquad \sin\theta = \frac{y}{r} \qquad \cos\theta = \frac{x}{r}$$

$$\sin\theta = \frac{y}{15}$$

$$\cos\theta\frac{d\theta}{dt} = \frac{1}{15}\cdot\frac{dy}{dt} \Rightarrow \frac{d\theta}{dt} = \left(\frac{1}{15\cos\theta}\right)\left(\frac{dy}{dt}\right)$$

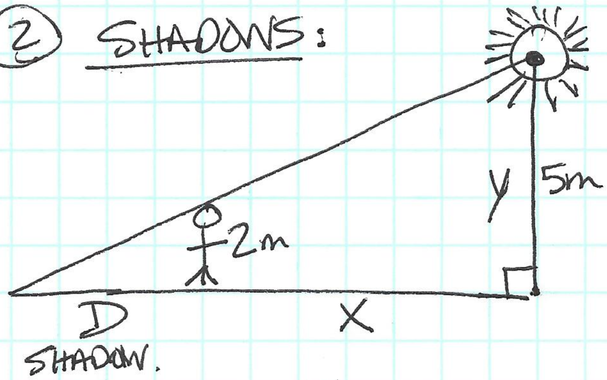
$$\frac{d\theta}{dt} = \left(\frac{1}{\cancel{15}}\right)\left(\frac{\cancel{15}}{5}\right)(-2)$$

$$= \frac{-2}{5}\ \frac{ft}{sec}$$

ATQ: The angle of elevation is decreasing at a rate of 2/5 rad/sec. when $x = 5$ft.

6 RELATED RATES PROBLEMS

(2) SHADOWS:



SIMILAR △s.

$$\frac{2}{D} = \frac{5}{D+x}$$

$$5D = 2D + 2x$$

$$3D = 2x$$

$$\frac{3}{2}D = x \quad \text{or} \quad D = \frac{2}{3}x$$

$\frac{dD}{dt} = ?$ Will be negative b/c shadow decreases as x decreases.

$\frac{dx}{dt} = -2\frac{m}{sec}$ will be negative b/c distance to street lamp is decreasing.

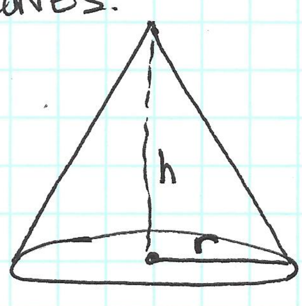
$$\frac{dD}{dt} = \frac{2}{3}\frac{dx}{dt}$$

$$\frac{dD}{dt} = \frac{2}{3}(-2)$$

$$\frac{dD}{dt} = \frac{-4}{3}\frac{m}{sec}.$$

ATQ: The length of the shadow is decreasing at a rate of $\frac{4}{3}\frac{m}{sec}$.

(3) CONES.



$$V = \frac{\pi}{3}r^2h \longrightarrow \frac{dV}{dt} = \frac{\pi}{3}\left(2rh\frac{dr}{dt} + r^2\frac{dh}{dt}\right)$$

$$h = 3r$$

$$\frac{dh}{dt} = 3\frac{dr}{dt}$$

$$\text{or} \quad \frac{1}{3}\frac{dh}{dt} = \frac{dr}{dt}$$

When $r = 20 \therefore h = 60$ ft.
& $\frac{dV}{dt} = 60\frac{ft^3}{sec}$ → subst.

FIND $\frac{dh}{dt} = ?$

$$\frac{3}{\pi}\frac{dV}{dt} = 2rh\left(\frac{1}{3}\frac{dh}{dt}\right) + r^2\frac{dh}{dt}$$

$$\frac{\left(\frac{3}{\pi}\right)\left(\frac{dV}{dt}\right)}{\left(\frac{2rh}{3} + r^2\right)} = \frac{dh}{dt}$$

$$\frac{3\left(\frac{dV}{dt}\right)}{\pi\left(\frac{2}{3}rh + r^2\right)} = \frac{dh}{dt}$$

$$\frac{3(60)}{\pi\left(\frac{2}{3}\cdot 20\cdot 60 + 20^2\right)}$$

$$\frac{3(60)}{\pi\left(2(20^2) + 20^2\right)}$$

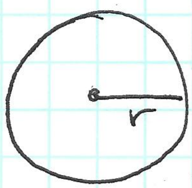
$$\frac{3(60)}{\pi(3)(20^2)} = \frac{60}{400\pi} = \frac{3}{20\pi}$$

$$\frac{dh}{dt} = \frac{60}{400\pi} = \frac{3}{20\pi}\frac{ft}{sec}$$

ATQ: The height is increasing at a rate of $3/(20\pi)$ ft/sec. when radius = 20 ft.

6 RELATED RATES PROBLEMS.

(4) CIRCLES:



$$A = \pi r^2 \qquad \frac{dr}{dt} = {}^{+}2\ \frac{ft}{sec} \qquad \text{Find } \frac{dA}{dt} = ?$$

$$A = 100 = \pi r^2$$

$$r^2 = \frac{100}{\pi}$$

$$r = \frac{10}{\sqrt{\pi}}$$

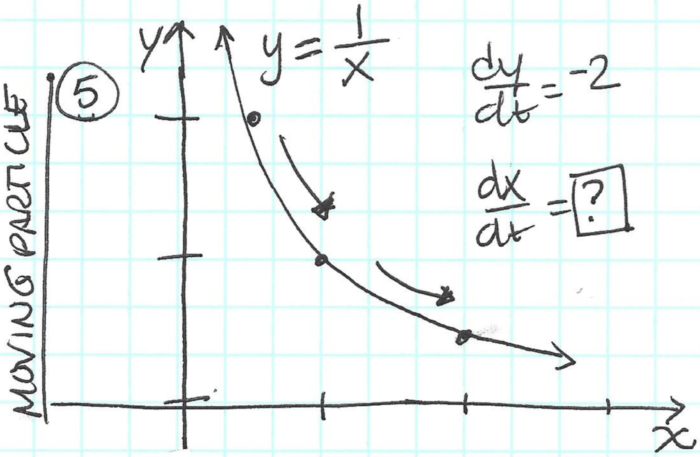
$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$$

$$\frac{dA}{dt} = 2\pi\left(\frac{10}{\sqrt{\pi}}\right)(2)$$

$$\frac{dA}{dt} = 40\sqrt{\pi}\ \frac{ft}{sec}$$

ATQ: The Area of the circle is increasing at a rate of $40\sqrt{\pi}\ \frac{ft}{sec}$ when Area is 100 ft^2.

(5) MOVING PARTICLE



$$\frac{dy}{dt} = -2$$

$$\frac{dx}{dt} = \boxed{?}$$

$$y = \frac{1}{x} \qquad \frac{dy}{dt} = \frac{-1}{x^2}\frac{dx}{dt}$$

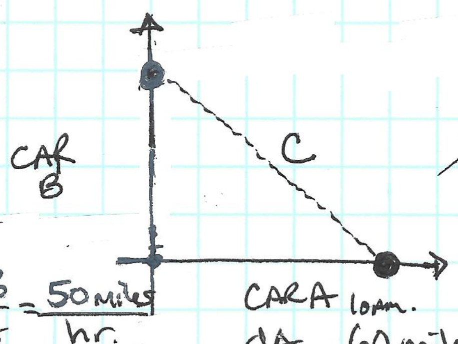
$$-x^2\frac{dy}{dt} = \frac{dx}{dt}$$

$$-(1)^2(-2) = \frac{dx}{dt}$$

$$\frac{dx}{dt} = 2\ \frac{\text{units}}{\text{time}}.$$

ATQ: The particle's x-position is increasing at a rate of 2 (linear units / unit of time) when its position is at (1,1).

(6) VEHICLE INTERSECTION



$\frac{dB}{dt} = \frac{50 \text{ miles}}{hr.}$

$\frac{dA}{dt} = \frac{60 \text{ miles}}{hr}$

$$C^2 = A^2 + B^2$$

$$2C\frac{dC}{dt} = 2(A)\frac{dA}{dt} + 2(B)\frac{dB}{dt}$$

$$\frac{dC}{dt} = \left(\frac{1}{C}\right)\left(A\frac{dA}{dt} + B\frac{dB}{dt}\right)$$

$$\frac{dC}{dt} = \frac{1}{130}\left(120(60) + 50(50)\right)$$

$$\frac{dC}{dt} = 74.615\ \frac{\text{miles}}{hr.}$$

at 12 noon: $A = 60x \therefore A = 120$

$B = 50(x-1) \therefore B = 50$

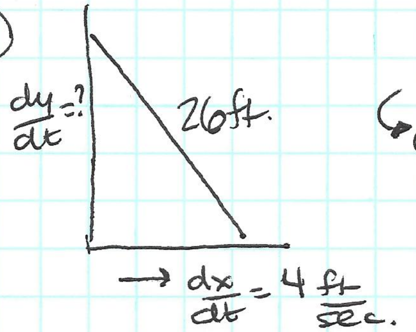
$C = 130$.

Pythag. Triple. 5-12-13

ATQ: At 12 noon the distance between car A & car B is increasing at a rate of 74.615 mph.

WS #2 RELATED RATES.

①



$x = 10 ft.$

$x^2 + y^2 = r^2$

$10^2 + y^2 = 26^2$

$5^2 + y^2 = 13^2$ Pythagorean Triple. $y = 12$

$y = 24 ft$

$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2r\frac{dr}{dt}$

$\frac{dy}{dt} = \left(r\frac{dr}{dt} - x\frac{dx}{dt}\right)\left(\frac{1}{y}\right)$

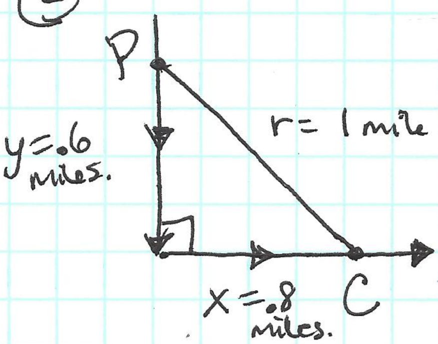
$\frac{dy}{dt} = \left(26(0) - 10(4)\right)\left(\frac{1}{24}\right)$

$\frac{dy}{dt} = -\frac{40}{24} \frac{ft}{sec} = -\frac{5}{3} \frac{ft}{sec}.$

ATQ

The ladder is sliding down the wall at a rate of $-\frac{5}{3} \frac{ft}{sec}$ when the foot of the ladder is 10ft from the base of the wall.

②



$x^2 + y^2 = r^2$

$(.8)^2 + (.6)^2 = r^2$

$r = 1$

$\frac{dr}{dt} = 20 \frac{mile}{hr}$

$\frac{dy}{dt} = -60 \frac{mile}{hr}$

$\frac{dx}{dt} = \boxed{?} \frac{mile}{hr}$

$x\frac{dx}{dt} + y\frac{dy}{dt} = r\frac{dr}{dt}$

$\frac{dx}{dt} = \left(r\frac{dr}{dt} - y\frac{dy}{dt}\right)\left(\frac{1}{x}\right)$

$\frac{dx}{dt} = \left((1)(20) - (.6)(-60)\right)\left(\frac{1}{.8}\right)$

$\frac{dx}{dt} = (20 + 36)\left(\frac{1}{.8}\right)$

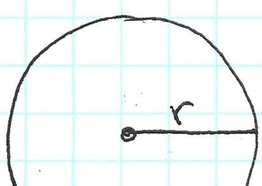
$\frac{dx}{dt} = (56)\left(\frac{1}{.8}\right)$

$\frac{dx}{dt} = 70 \frac{mile}{hr}$

ATQ

The car is moving away from the intersection at a rate of 70 miles/hr when the distance between the two cars is 1 mile.

③



$\frac{dr}{dt} = 0.01 \frac{cm}{sec}$

$r = 50$ cm.

$A = \pi r^2$

$\frac{dA}{dt} = 2\pi r \cdot \frac{dr}{dt}$

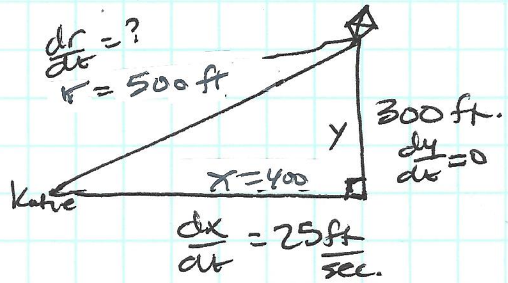
$\frac{dA}{dt} = 2\pi(50)(0.01)$

$\frac{dA}{dt} = \pi\ cm^2/sec$

ATQ

The Area of the metal plate is increasing at a rate of $\pi\ cm^2/sec$ when the radius is 50 cm.

(4)



$x^2 + y^2 = r^2$

$x^2 + 300^2 = 500^2$

$\therefore x = 400$

Find $\frac{dr}{dt}$ when $x = 500$ ft.

$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2r\frac{dr}{dt}$

$x\frac{dx}{dt} + y\frac{dy}{dt} = r\frac{dr}{dt}$

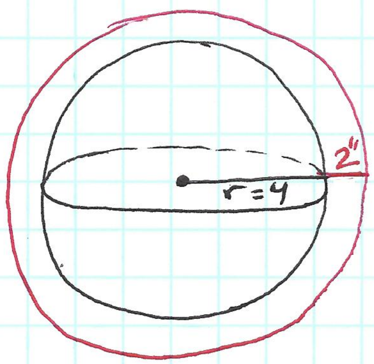
$\left(\frac{1}{r}\right)\left(x\frac{dx}{dt} + y\frac{dy}{dt}\right) = \frac{dr}{dt}$

$\left(\frac{1}{500}\right)\left((400)(25) + (300)(0)\right) = \frac{dr}{dt}$

$\frac{dr}{dt} = 20\ \frac{ft}{sec}$

ATQ:
Katie must let out the kite string at a rate of 20 ft/sec when the kite is 500 ft from her.

(5)



$\frac{dr}{dt} = ?$

$\frac{dV}{dt} = -10\ \frac{in^3}{min}$

When 2" thick $\therefore r = 6$

$V = \frac{4\pi}{3}r^3$

$\frac{dV}{dt} = 4\pi r^2\frac{dr}{dt}$

$\frac{dr}{dt} = \frac{\frac{dV}{dt}}{4\pi r^2} = \frac{-10}{4\pi(6)^2} = \frac{-5}{72\pi}\ \frac{in}{min}$

a) ATQ: When ice is 2 inches thick, the thickness of ice is decreasing at a rate of $\frac{5}{72\pi}$ inches/minute.

b) $SA = 4\pi r^2$

$\frac{d(SA)}{dt} = 8\pi r\frac{dr}{dt}$

$= 8\pi(6)\left(\frac{-5}{72\pi}\right)$

$= -\frac{8\cdot 6\cdot 5\pi}{72\cdot 6\cdot \pi} = \frac{-10}{3}\ \frac{in^2}{min}$

ATQ: When the ice is 2 inches thick, the surface area of ice is decreasing at a rate of $\frac{-10}{3}$ square inches/minute.