2.6 Continuity and Differentiability—Student Notes HH6ed

Definition: A function $f\left(x\right)$ is **continuous at a number *a*** if

$$\lim_{x\to a} f(x) = f(a)$$

This definition implicitly requires three things to be continuous at $x = a$:

1. $f\left(a\right)$ exists (that is, a is in the domain of $f\left(x\right)$)
2. $\lim_{x\to a} f(x)$ exists (so $f\left(x\right)$ must be defined on an open interval that contains a)
3. $\lim_{x\to a} f(x) = f(a)$

There are 3 types of Discontinuity:

1. Removable Discontinuity: A limit exists, but there is a hole at the value.

2. Non-removable (or Jump) Discontinuity : A limit does not exist at the value.

3. Infinite Discontinuity: There is a vertical asymptote at the value. The limit from the left and right is ∞ or $-\infty$

Examples:

1. Where are each of the following functions discontinuous? State the type of discontinuity.

a. $f(x) = \dfrac{x^2 - x - 2}{x-2}$

b. $f(x) = \begin{cases} \dfrac{1}{x^2}, & \text{if } x \neq 0 \\ 1, & \text{if } x = 0 \end{cases}$

c. $f(x) = \begin{cases} \dfrac{x^2 - x - 2}{x-2}, & \text{if } x \neq 2 \\ 1, & \text{if } x = 2 \end{cases}$

d. $f(x) = x$

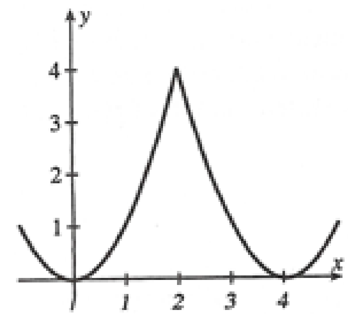
<u>Definition</u> A function $f(x)$ is **differentiable at *a*** if $f'(a)$ exists. It is **differentiable on an open interval** (a, b) [or (a,∞) or $(-\infty,a)$ or $(-\infty,\infty)$] if it is differentiable at every number in the interval.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

There are 3 common ways for a function to fail to be differentiable at a point

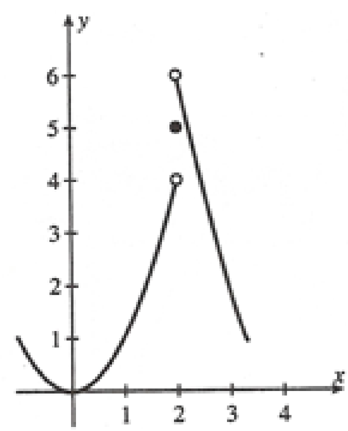
a. The graph has a **sharp point or cusp**.

<u>Example:</u> $f(x) = \begin{cases} x^2 \ \ if \ \ x \le 2 \\ (x-2)^2 \ \ if \ \ x > 2 \end{cases}$



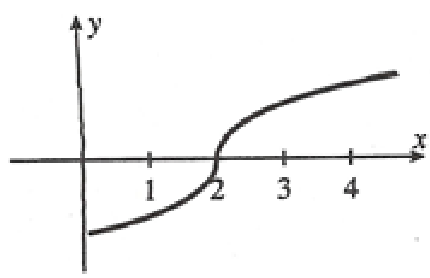
b. The function is **discontinuous**. (break, hole or asymptote)

<u>Example:</u> $f(x) = \begin{cases} x^2 \ \ if \ \ x < 2 \\ 5 \ \ if \ \ x = 2 \\ 10 - x^2 \ \ if \ \ x > 2 \end{cases}$



c. The graph has a **vertical tangent line**.

<u>Example:</u> $f(x) = \sqrt[3]{x-2}$



Theorem: If $f(x)$ is differentiable at a point $x = c$, then $f(x)$ is continuous at c. The converse is **false**.

* differentiability $\Rightarrow$ continuity $\Rightarrow$ limit

Examples:

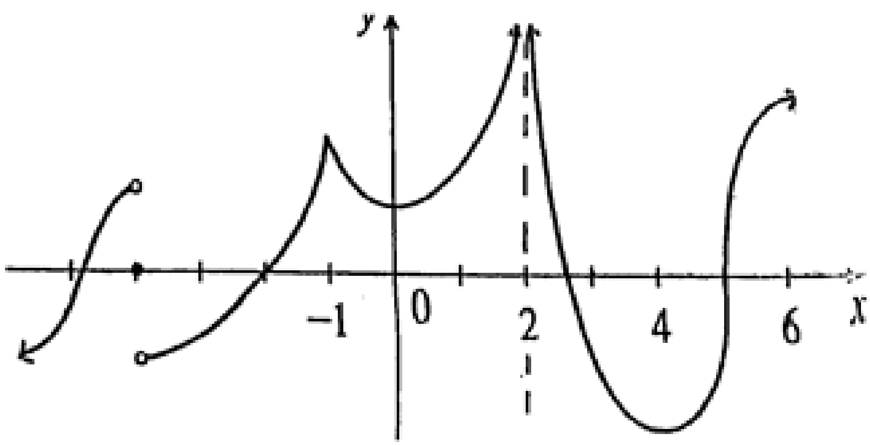
2. Is the absolute value function differentiable at $x = 0$? Explain.

3. Is $f(x) = x^{\frac{1}{3}}$ differentiable at $x = 0$? Explain.

4. Is $f(x) = (x-1)^{\frac{2}{3}}$ differentiable at $x = 1$? Explain.

5. Is $f(x) = \dfrac{x^2 - 5x + 6}{x - 3}$ differentiable at $x = 3$?

6. Refer to the figure at the right. Complete the following table indicating at which values on the open interval (-6, 6), the given function, *f*, fails to be continuous and/or differentiable.



Domain value	Continuous? (yes or no)	If no, why?	Differentiable? (yes or no)	If no, why
1.				
2.				
3.				
4.				