AP Calculus

Areas and Volumes

Student Handout

2016-2017 EDITION

Use the following link or scan the QR code to complete the evaluation for the Study Session https://www.surveymonkey.com/r/S_SSS

Areas and Volumes

Students should be able to:

- Apply definite integrals to problems involving area and volume.
- Set up and evaluate an integral expression to calculate the area of a region between two curves.
- Set up and evaluate an integral expression to calculate the volume of solids with known cross sections, including discs and washers.
- Recognize when multiple integrals must be used to calculate area or volume.
- Translate the information in a definite integral into the limit of a related Riemann sum.

Multiple Choice

1. (calculator not allowed)
The region enclosed by the *x*-axis, the line $x = 3$, and the curve $y = \sqrt{x}$ is rotated about the *x*-axis. What is the volume of the solid generated?

(A) 3π
(B) $2\sqrt{3}\pi$
(C) $\frac{9}{2}\pi$
(D) 9π
(E) $\frac{36\sqrt{3}}{5}\pi$

2. (calculator not allowed)
What is the area of the region in the first quadrant bounded by the graph of $y = e^{\frac{x}{2}}$ and the line $x = 2$?

(A) $2e - 2$
(B) $2e$
(C) $\frac{e}{2} - 1$
(D) $\frac{e-1}{2}$
(E) $e - 1$

3. (calculator allowed)
What is the area enclosed by the curves $y = x^3 - 8x^2 + 18x - 5$ and $y = x + 5$?

(A) 10.667
(B) 11.833
(C) 14.583
(D) 21.333
(E) 32

4. (calculator not allowed)

The region bounded by the x-axis and the part of the graph of $y = \cos x$ between $x = -\frac{\pi}{2}$ and $x = \frac{\pi}{2}$ is separated into two regions by the line $x = k$. If the area of the region for $-\frac{\pi}{2} \le x \le k$ is three times the area of the region for $k \le x \le \frac{\pi}{2}$, then $k =$

(A) $\arcsin\left(\frac{1}{4}\right)$

(B) $\arcsin\left(\frac{1}{3}\right)$

(C) $\frac{\pi}{6}$

(D) $\frac{\pi}{4}$

(E) $\frac{\pi}{3}$

5. (calculator not allowed)

Let R be the region in the first quadrant bounded above by the graph of $y = \sqrt{x}$ and below by the graph of $y = x^2$. R is the base of a solid whose cross sections perpendicular to the $y-$axis are squares. Which of the following gives the volume of the solid?

(A) $\int_0^1 \left(\sqrt{x} - x^2\right)^2 dx$

(B) $\int_0^1 \left(x - x^4\right) dx$

(C) $\int_0^1 \left(\sqrt{y} - y^2\right)^2 dy$

(D) $\int_0^1 \left(\sqrt{y} - y^2\right) dy$

6. (calculator allowed)

Let R be the region in the first and second quadrants bounded above by the graph of $y = \frac{20}{1+x^2}$ and below by the horizontal line $y = 2$. R is the base of a solid whose cross sections perpendicular to the $x-$axis are semicircles. What is the volume of the solid?

(A) 29.815

(B) 174.268

(C) 348.537

(D) 443.771

7. (calculator not allowed)

The functions f and g are given by $f(x) = 2\sqrt{x}$ and $g(x) = x - 3$. Let R be the region bounded by the $x-$axis and the graphs of f and g. The graphs of f and g intersect in the first quadrant at the point $(9, 6)$. Which of the following gives the volume of the solid generated when R is revolved about the $x-$axis?

(A) $4\pi\int_0^3 x\,dx + \pi\int_3^9 \left(4x - (x-3)^2\right)\,dx$

(B) $4\pi\int_0^3 x\,dx + \pi\int_3^9 \left(2\sqrt{x} - (x-3)\right)^2\,dx$

(C) $\pi\int_0^9 \left(2\sqrt{x} - (x-3)\right)^2\,dx$

(D) $\pi\int_0^9 \left(4x - (x-3)^2\right)\,dx$

8. (calculator not allowed)

The function f is given by $f(x) = \ln x$. Which of the following limits is equal to the area between the graph of $f(x)$ and the $x-$axis from $x = 1$ to $x = 3$?

(A) $\lim_{n\to\infty}\sum_{k=1}^{n} \ln\left(1 + \frac{2k}{n}\right)\frac{2}{n}$

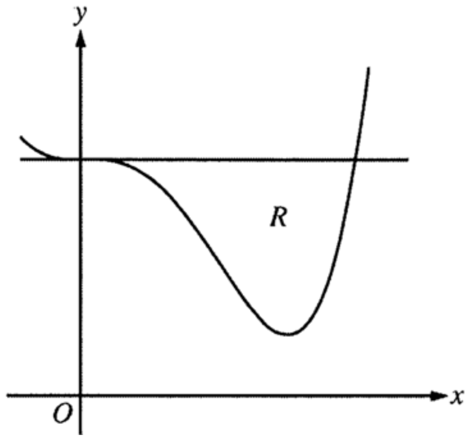
(B) $\lim_{n\to\infty}\sum_{k=1}^{n} \ln\left(1 + \frac{2k}{n}\cdot\frac{2}{n}\right)$

(C) $\lim_{n\to\infty}\sum_{k=1}^{n} \ln\left(1 + \frac{2k}{n}\right)\frac{1}{n}$

(D) $\lim_{n\to\infty}\sum_{k=1}^{n} \ln\left(\frac{2k}{n}\right)\frac{2}{n}$

Free Response

9. (calculator allowed)



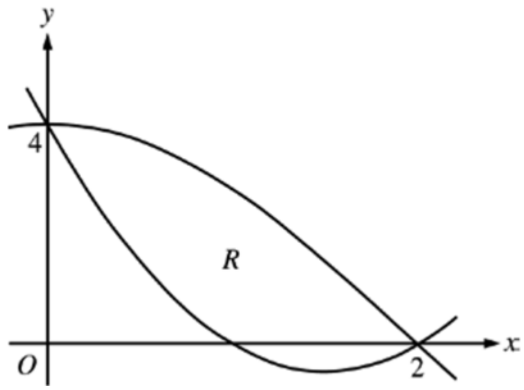
Let R be the region enclosed by the graph of $f(x) = x^4 - 2.3x^3 + 4$ and the horizontal line $y = 4$, as shown in the figure above.

(a) Find the volume of the solid generated when R is rotated about the horizontal line $y = -2$.

(b) Region R is the base of a solid. For this solid, each cross section perpendicular to the x-axis is an isosceles right triangle with a leg in R. Find the volume of the solid.

(c) The vertical line $x = k$ divides R into two regions with equal areas. Write, but do not solve, an equation involving integral expressions whose solution give the value of k.

10. (calculator not allowed)



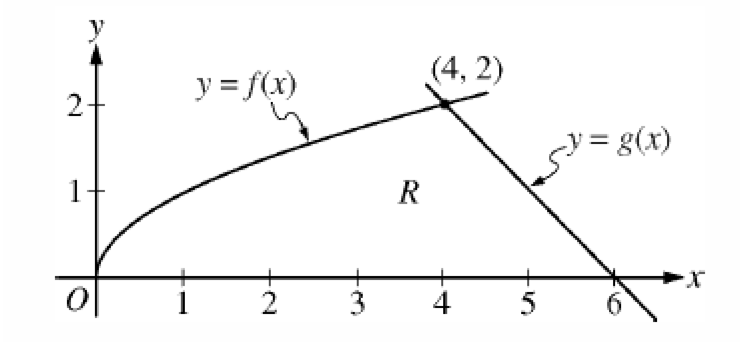
Let $f(x) = 2x^2 - 6x + 4$ and $g(x) = 4\cos\left(\frac{1}{4}\pi x\right)$. Let R be the region bounded by the graphs of f and g, as shown in the figure above.

(a) Find the area of R.

(b) Write, but do not evaluate, an integral expression that gives the volume of the solid generated when R is rotated about the horizontal line $y = 4$.

(c) The region R is the base of a solid. For this solid, each cross section perpendicular to the x-axis is a square. Write, but do not evaluate, an integral expression that gives the volume of the solid.

11. (calculator not allowed)

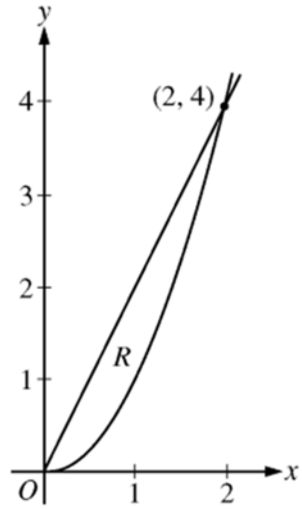


The functions f and g are given by $f(x) = \sqrt{x}$ and $g(x) = 6 - x$. Let R be the region bounded by the x-axis and the graphs of f and g, as shown in the figure above.

(a) Find the area of R.

(b) The region R is the base of a solid. For each y, where $0 \le y \le 2$, the cross section of the solid taken perpendicular to the y-axis is a rectangle whose base lies in R and whose height is $2y$. Write, but do not evaluate, an integral expression that gives the volume of the solid.

12. (calculator not allowed)



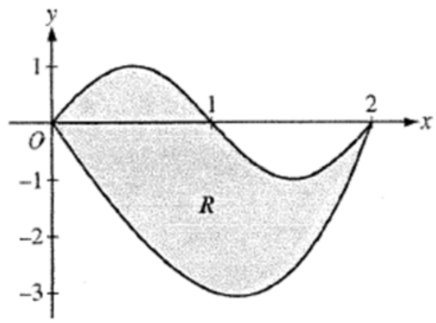
Let R be the region in the first quadrant enclosed by the graphs of $y = 2x$ and $y = x^2$, as shown in the figure above.

(a) Find the area of R.

(b) The region R is the base of a solid. For this solid, at each x the cross section perpendicular to the $x-$axis has area $A(x) = \sin\left(\frac{\pi}{2}x\right)$. Find the volume of this solid.

(c) Another solid has the same base R. For this solid, the cross sections perpendicular to the $y-$axis are squares. Write, but do not evaluate, an integral expression for the volume of the solid.

13. (calculator allowed)



Let R be the region bounded by the graphs of $y = \sin(\pi x)$ and $y = x^3 - 4x$, as shown in the figure above.

(a) Find the area of R.

(b) The horizontal line $y = -2$ splits the region R into two parts. Write, but do not evaluate, an integral expression for the area of the part of R that is below this horizontal line.

(c) The region R is the base of a solid. For this solid, each cross section perpendicular to the x-axis is an equilateral triangle. Find the volume of this solid.

(d) The region R models the surface of a small pond. At all points in R at a distance x from the y-axis, the depth of the water is given by $h(x) = 3 - x$. Find the volume of water in the pond.

Area and Volume Reference Page

Area between two curves

- Sketch the region and determine the points of intersection.
- Draw a small strip either as dx or dy slicing.
- Use the following templates to set up a definite integral:

dx slicing: $A = \int_{left\,x}^{right\,x} \left(y_{top} - y_{bottom} \right) dx$ where y_{top} and y_{bottom} are written in terms of x.

dy slicing: $A = \int_{bottom\,y}^{top\,y} \left(x_{right} - x_{left} \right) dy$ where x_{right} and x_{left} are written in terms of y.

Volume of a Solid with a Known Cross-Section

- Sketch the region and draw a representative rectangle to be used in determining whether setting up with respect to dx or dy.
- Determine the slicing direction then find the volume of the slice which will be the area of the "face" times the "thickness".
- Determine the total volume by summing up the slices using a definite integral.
- Use the following templates to set up a definite integral.

dx slicing: $V = \int_{left\,x}^{right\,x} A(x)\,dx$ where *A(x)* is the area of the face written in terms of x.

dy slicing: $V = \int_{bottom\,y}^{top\,y} A(y)\,dy$ *where A(y)* is the area of the face written in terms of y.

- Useful formulas to memorize:

Area of an equilateral triangle: $A = \frac{\sqrt{3}}{4} s^2$

Area of a semi-circle in terms of its diameter: $A = \frac{\pi}{8} d^2$

Volume of a Solid of Revolution

- Sketch the region to be revolved and a representative rectangle whose width can be used to determine whether integrating with dx or dy.
- Set up a definite integral after determining whether the slicing uses dx or dy so that the slicing is perpendicular to the axis of revolution.
- Identify the outside radius and the inside radius and use the appropriate template from below:

- *dx slicing:* $V = \pi \int_{left\,x}^{right\,x} \left((outside\,radius)^2 - (inside\,radius)^2 \right) dx$

  where the outside and inside radii are written in terms of x.

  dy slicing: $V = \pi \int_{bottom\,y}^{top\,y} \left((outside\,radius)^2 - (inside\,radius)^2 \right) dy$

  where the outside and inside radii are written in terms of y.