NATIONAL MATH + SCIENCE INITIATIVE

# AP Calculus

# AB Integrals and Their Applications

## Student Handout

**2016-2017 EDITION**

Use the following link or scan the QR code to complete the evaluation for the Study Session https://www.surveymonkey.com/r/S_SSS

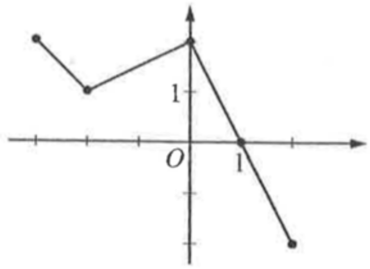
# Integrals and Their Applications (AP Calculus AB)

**Students should be able to**:

- recognize antiderivatives of the basic functions using differentiation rules as the foundation
- calculate antiderivatives of functions using algebraic manipulation techniques such as long division, completing the square and *u*-substitution
- interpret the definite integral as the limit of a Riemann sum, and also express the limit of a Riemann sum in integral notation
- calculate a definite integral using the properties and geometric interpretations of definite integrals
- use the definite integral to solve problems in various contexts
- recognize that the definite integral of the rate of change of a quantity over an interval gives the net change of that quantity over that interval
- extend the definition of the definite integral to functions with removable or jump discontinuities
- apply definite integrals to problems involving the average value of a function, motion, area and volume
- use integrals to solve separable differential equations

Multiple Choice

1. (calculator not allowed)




Graph of $f$

The graph of a piecewise linear function $f(x)$, for $-3 \le x \le 2$, is shown above. What is the value of $\int_{-2}^{2}(f(x)+2)\,dx$ ?

(A) 5
(B) 6.5
(C) 11
(D) 12.5

2. (calculator not allowed)

Let $f(x)$ be a continuous function. Using the substitution $u = 3x+1$, the integral

$$\int_{1}^{4} f(3x+1)dx$$

is equal to which of the following?

(A) $\int_{1}^{4} f(u)du$

(B) $\frac{1}{3}\int_{1}^{4} f(u)du$

(C) $3\int_{4}^{13} f(u)du$

(D) $\frac{1}{3}\int_{4}^{13} f(u)du$

3. (calculator not allowed)

$$\int \frac{1}{x^2}\,dx =$$

(A) $\ln x^2 + C$
(B) $-\ln x^2 + C$
(C) $x^{-1} + C$
(D) $-x^{-1} + C$
(E) $-2x^{-3} + C$

4. (calculator not allowed)

Which of the following integral expressions is equal to $\lim_{n\to\infty}\sum_{k=1}^{n}\left(\left(1+\frac{2k}{n}\right)^2\cdot\frac{1}{n}\right)$ ?

(A) $\int_0^1 (1+2x)^2\,dx$

(B) $\int_0^2 (1+x)^2\,dx$

(C) $\int_1^3 x^2\,dx$

(D) $\frac{1}{2}\int_0^2 x^2\,dx$

5. (calculator not allowed)

$$f(x)=\begin{cases} x & \text{for } x<2 \\ 3 & \text{for } x\geq 2 \end{cases}$$

If $f$ is the function defined above, then $\int_{-1}^{4} f(x)dx$ is

(A) $\frac{9}{2}$

(B) $\frac{15}{2}$

(C) $\frac{17}{2}$

(D) undefined

6. (calculator not allowed)

$$\int_0^3 \frac{x^2+4x+5}{x+3}\,dx =$$

(A) $\frac{84}{63}$
(B) $3+2\ln 2$
(C) $\frac{15}{2}+2\ln 2$
(D) $\frac{15}{2}+2\ln 3$

7. (calculator not allowed)

$$\int \frac{1}{x^2+6x+13}\,dx =$$

(A) $\frac{1}{2}\arctan\frac{(x+3)}{2}+C$

(B) $\dfrac{1}{\frac{x^3}{3}+3x^2+13x}+C$

(C) $\ln\left|x^2+6x+13\right|+C$

(D) $2\arctan\frac{(x+3)}{2}+C$

8. (calculator not allowed)
At time $t$, a population of bacteria grows at a rate of $5e^{0.2t}+4t$ grams per day, where $t$ is measured in days. By how many grams has the population grown from time $t=0$ to $t=10$ days?

(A) $5e^2+40$
(B) $5e^2+195$
(C) $25e^2+175$
(D) $25e^2+375$

9. (calculator not allowed)

Which of the following limits is equal to $\int_3^7 x^3 dx$ ?

(A) $\lim_{n\to\infty}\sum_{k=1}^{n}\left(\left(3+\frac{k}{n}\right)^3\cdot\frac{1}{n}\right)$

(B) $\lim_{n\to\infty}\sum_{k=1}^{n}\left(\left(3+\frac{k}{n}\right)^3\cdot\frac{4}{n}\right)$

(C) $\lim_{n\to\infty}\sum_{k=1}^{n}\left(\left(3+\frac{4k}{n}\right)^3\cdot\frac{1}{n}\right)$

(D) $\lim_{n\to\infty}\sum_{k=1}^{n}\left(\left(3+\frac{4k}{n}\right)^3\cdot\frac{4}{n}\right)$

10. (calculator not allowed)

$\int e^x \cos\left(e^x+1\right)dx =$

(A) $\sin\left(e^x+1\right)+C$

(B) $e^x\sin\left(e^x+1\right)+C$

(C) $e^x\sin\left(e^x+x\right)+C$

(D) $\frac{1}{2}\cos^2\left(e^x+1\right)+C$

11. (calculator not allowed)

Using the substitution $u=\sqrt{x}$, the integral $\int_1^9 \frac{\sin\sqrt{x}}{\sqrt{x}}dx$ is equal to which of the following?

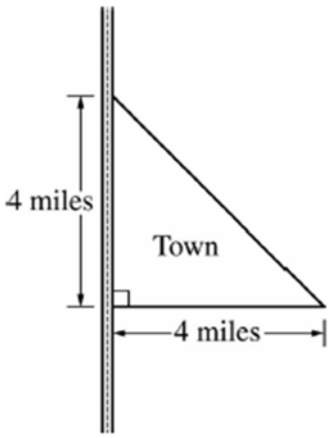
(A) $\frac{1}{2}\int_1^3 \sin u\, du$

(B) $2\int_1^3 \frac{\sin u}{u}\, du$

(C) $2\int_1^3 \sin u\, du$

(D) $2\int_1^9 \sin u\, du$

12. (calculator not allowed)




The right triangle shown in the figure represents the boundary of a town that is bordered by a highway. The population density of the town at a distance of $x$ miles from the highway is modeled by $D(x)=\sqrt{x+1}$, where $D(x)$ is measured in thousands of people per square mile. According to the model, which of the following expressions gives the total population, in thousands, of the town?

(A) $\int_0^4 \sqrt{x+1}\,dx$

(B) $\int_0^4 8\sqrt{x+1}\,dx$

(C) $\int_0^4 x\sqrt{x+1}\,dx$

(D) $\int_0^4 (4-x)\sqrt{x+1}\,dx$

13. (calculator not allowed)

$$f(x)=\begin{cases}\dfrac{|x-1|}{x-1}, & x\neq 1\\ 1, & x=1\end{cases}$$

If $f$ is the function defined above, then $\int_{-1}^{4} f(x)dx$ is

(A) 1
(B) 2
(C) 5
(D) nonexistent

14. (calculator not allowed)

$$\int_1^2 \frac{x^2+6x+6}{x+1}\,dx =$$

(A) $1+\ln\frac{3}{2}$

(B) $6.5$

(C) $6.5+\ln\frac{3}{2}$

(D) $6.5+\ln 6$

15. (calculator not allowed)

$$\int \frac{x}{x^2-4}\,dx =$$

(A) $\frac{-1}{4\left(x^2-4\right)^2}+C$

(B) $\frac{1}{2\left(x^2-4\right)}+C$

(C) $\frac{1}{2}\ln\left|x^2-4\right|+C$

(D) $2\ln\left|x^2-4\right|+C$

(E) $\frac{1}{2}\arctan\left(\frac{x}{2}\right)+C$

16. (calculator not allowed)

$$\int_1^e \frac{x^2-1}{x}\,dx =$$

(A) $e-\frac{1}{e}$

(B) $e^2-e$

(C) $\frac{e^2}{2}-e+\frac{1}{2}$

(D) $e^2-2$

(E) $\frac{e^2}{2}-\frac{3}{2}$

17. (calculator not allowed)

$\int_0^1 \frac{\arctan x}{1+x^2}\, dx = \;=$

(A) $\frac{\pi}{4}$

(B) $\frac{\pi^2}{32}$

(C) $\frac{\pi^2}{16}$

(D) 1

18. (calculator allowed)

A rain barrel collects water off the roof of a house during three hours of heavy rainfall. The height of the water in the barrel increases at the rate of $r(t) = 4t^3e^{-1.5t}$ feet per hour where $t$ is the time in hours since the rain began. At time $t = 1$ hour, the height of water is 0.75 foot. What is the height of water in the barrel at time $t = 2$ hours?

(A) 1.361 ft
(B) 1.500 ft
(C) 1.672 ft
(D) 2.111 ft

19. (calculator allowed)

The function g is continuous on the closed interval [2, 10]. If $\int_2^{10} g(x)dx = 63$ and $\int_{10}^{5} \frac{1}{2} g(x)dx = -16$, then $\int_2^5 2g(x)dx =$

(A) 31
(B) 62
(C) 95
(D) 190

20. (calculator not allowed)
A pizza, heated to a temperature of 350 degrees Fahrenheit ( $^0$F), is taken out of an oven and placed in a 75$^o$F room at time $t = 0$ minutes. The temperature of the pizza is changing at a rate of $-110e^{-0.4t}$ degrees Fahrenheit per minute. Which of the following is the best interpretation of $\int_0^{30} -110e^{-0.4t}\,dt$ ?
(A) The average temperature of the pizza, in degrees Fahrenheit, over the 30 minutes time interval after it was taken out of the oven.
(B) Temperature of the pizza, in degrees Fahrenheit, 30 minutes after it was taken out of the oven.
(C) The rate at which the temperature of the pizza is changing, in degrees Fahrenheit per minute, 30 minutes after it was taken out of the oven.
(D) The change in the temperature of the pizza, in degrees Fahrenheit, over the 30 minutes time interval after it was taken out of the oven.

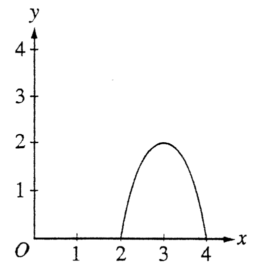
21. (calculator allowed)
What is the average value of $y = \dfrac{\cos x}{x^2 + x + 2}$ on the closed interval [-1, 3]?

(A) -0.085
(B) 0.090
(C) 0.183
(D) 0.244
(E) 0.732

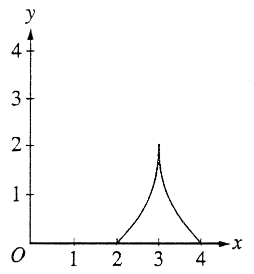
22. (calculator allowed)

On the closed interval $[2,4]$, which of the following could be the graph of a function $f$ with the property that $\frac{1}{4-2}\int_2^4 f(t)\,dt = 1$?

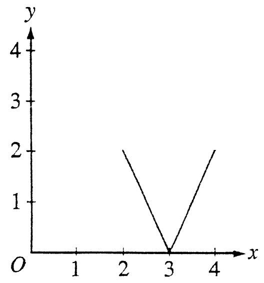
(A)



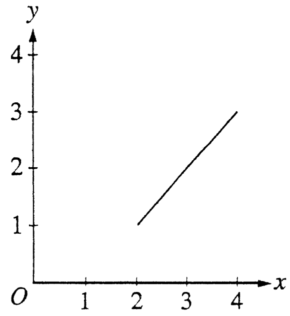

(B)



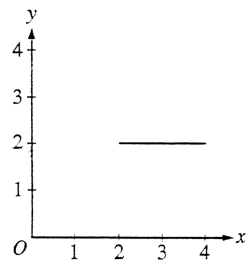

(C)




(D)




(E)




23. (calculator not allowed)

If $f$ is a continuous function and if $F'(x) = f(x)$ for all real numbers $x$, then $\int_1^3 f(2x)\,dx =$

(A) $2F(3) - 2F(1)$

(B) $\frac{1}{2}F(3) - \frac{1}{2}F(1)$

(C) $2F(6) - 2F(2)$

(D) $F(6) - F(2)$

(E) $\frac{1}{2}F(6) - \frac{1}{2}F(2)$

Free Response

24. (calculator not allowed)

| $x$ | $-2$ | $-2 < x < -1$ | $-1$ | $-1 < x < 1$ | $1$ | $1 < x < 3$ | $3$ |
|---|---|---|---|---|---|---|---|
| $f(x)$ | 12 | Positive | 8 | Positive | 2 | Positive | 7 |
| $f'(x)$ | $-5$ | Negative | 0 | Negative | 0 | Positive | $\frac{1}{2}$ |
| $g(x)$ | $-1$ | Negative | 0 | Positive | 3 | Positive | 1 |
| $g'(x)$ | 2 | Positive | $\frac{3}{2}$ | Positive | 0 | Negative | $-2$ |

The twice differentiable functions $f$ and $g$ are defined for all real numbers $x$. Values of $f$, $f'$, $g$ and $g'$ for various values of $x$ are given in the table above.

(d) Evaluate $\int_{-2}^{3} f'(g(x))\, g'(x)\, dx$.

(e) Evaluate $\int_{-2}^{-1} (2f'(x)+3)\, dx$

25. (calculator allowed)
On a certain workday, the rate, in tons per hour, at which unprocessed gravel arrives at a gravel processing plant is modeled by $G(t) = 90 + 45\cos\left(\frac{t^2}{18}\right)$, where $t$ is measured in hours and $0 \le t \le 8$. At the beginning of the workday $(t = 0)$, the plant has 500 tons of unprocessed gravel. During the hours of operation, $0 \le t \le 8$, the plant processes gravel at a constant rate of 100 tons per hour.

(b) Find the total amount of unprocessed gravel that arrives at the plant during the hours of operation on this workday.

(d) What is the maximum amount of unprocessed gravel at the plant during the hours of operation on this workday? Justify your answer.

26. (calculator not allowed)

    Find the average value of $f(x) = 4\cos\left(\frac{\pi}{4}x\right)$ on the interval $[0,2]$.

27. (calculator allowed)
    A tank contains 125 gallons of heating oil at time $t = 0$. During the time interval $0 \le t \le 12$ hours, heating oil is pumped into the tank at the rate

    $$H(t) = 2 + \frac{10}{\left(1 + \ln(t+1)\right)} \text{ gallons per hour.}$$

    During the same time interval, heating oil is removed form the tank at the rate

    $$R(t) = 12\sin\left(\frac{t^2}{47}\right) \text{ gallons per hour.}$$

    (a) How many gallons of heating oil are pumped into the tank during the time interval $0 \le t \le 12$ hours?

    (c) How many gallons of heating oil are in the tank at time $t = 12$ hours?

    (d) At what time $t$, for $0 \le t \le 12$, is the volume of heating oil in the tank the least? Show the analysis that leads to your conclusion.

28. (calculator allowed)

Grass clippings are placed in a bin, where they decompose. For $0 \le t \le 30$, the amount of grass clippings remaining in the bin is modeled by $A(t) = 6.687(0.931)^t$, where $A(t)$ is measured in pounds and $t$ is measured in days.

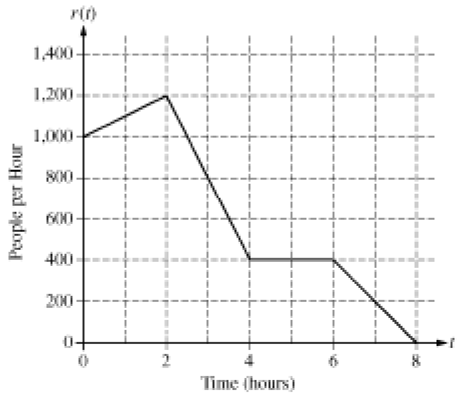
(c) Find the time $t$ for which the amount of grass clippings in the bin is equal to the average amount of grass clippings in the bin over the interval $0 \le t \le 30$.

29. (calculator not allowed)

The function $f$ is defined by $f(x) = \sqrt{25 - x^2}$ for $-5 \le x \le 5$.

(d) Find the value of $\int_0^5 x\sqrt{25 - x^2}\,dx$.

30. (calculator allowed)



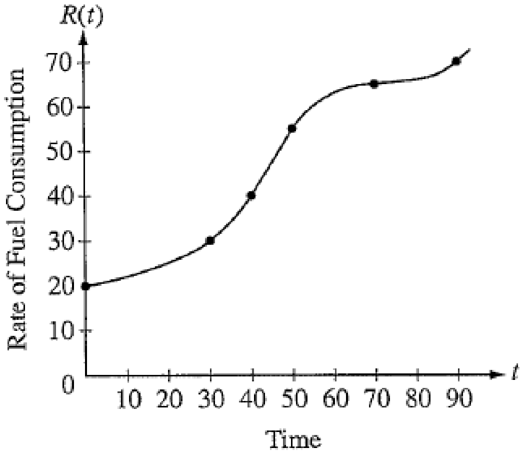

There are 700 people in line for a popular amusement-park ride when the ride begins operation in the morning. Once it begins operation, the ride accepts passengers until the park closes 8 hours later. While there is a line, people move onto the ride at a rate of 800 people per hour. The graph above shows the rate, $r(t)$, at which people arrive at the ride throughout the day. Time $t$ is measured in hours from the time the ride begins operation.

(a) How many people arrive at the ride between $t = 0$ and $t = 3$?
Show the computations that lead to your answer.

(d) Write, but do not solve, an equation involving an integral expression of $r$ whose solutions gives the earliest time $t$ at which there is no longer a line for the ride.

31. (calculator allowed)




| $t$ (minutes) | $R(t)$ (gallons per minute) |
|---|---|
| 0 | 20 |
| 30 | 30 |
| 40 | 40 |
| 50 | 55 |
| 70 | 65 |
| 90 | 70 |

The rate of fuel consumption, in gallons per minute, recorded during an airplane flight is given by a twice-differentiable and strictly increasing function $R$ of time $t$. The graph of $R$ and a table of selected values of $R(t)$, for the time interval $0 \le t \le 90$ minutes, are shown above.

(d) For $0 < b \le 90$ minutes, explain the meaning of $\int_0^b R(t)\,dt$ in terms of fuel consumption for the plane. Explain the meaning of $\frac{1}{b}\int_0^b R(t)\,dt$ in terms of fuel consumption for the plane. Indicate units of measure in both answers.

(e) Using correct units of measure explain the meaning of $\frac{R(90)-R(0)}{90-0}$ in the context of the problem.

32. (calculator not allowed)

Let $f$ be a continuous function defined by $f(x) = \begin{cases} 1-2\sin x & \text{for} \quad x \le 0 \\ e^{-4x} & \text{for} \quad x > 0 \end{cases}$

(c) Find the average value of $f$ on the interval $[-1, 1]$.

# Integrals and Their Applications Reference Page

Basic Integration

$\int k\, f(u)\,du =$ ____________________

$\int [f(u) \pm g(u)]\,du =$ ________________

$\int du =$ __________________________

$\int u^n\,du =$ ______________________

$\int \frac{du}{u} =$ _______________________

$\int a^u\,du =$ _______________________

$\int e^u\,du =$ _______________________

Inverse Trigonometric

$\int \frac{du}{\sqrt{a^2 - u^2}} =$ _____________________

$\int \frac{du}{a^2 + u^2} =$ _______________________

Trigonometric Functions:

$\int \sin(u)\,du =$ _____________________

$\int \cos(u)\,du =$ _____________________

$\int \sec^2(u)\,du =$ __________________

$\int \csc^2(u)\,du =$ ____________________

$\int \sec(u)\tan(u)\,du =$ ________________

$\int \csc(u)\cot(u)\,du =$ ________________

Helpful to know:

$\int \tan(u)\,du = \int \frac{\sin u}{\cos u}\,du =$ ___________

$\int \cot(u)\,du = \int \frac{\cos u}{\sin u}\,du =$ ___________

Fundamental Theorem of Calculus:

$$\int_a^b f(x)dx = F(b) - F(a)$$

$$\int_a^b f'(x)dx = f(b) - f(a)$$

$$f(b) = f(a) + \int_a^b f'(x)dx$$

FTC in context:

$$\text{Current Amount} =$$

$$\text{Initial Amount} + \int_{time1}^{time2} (\text{"rate in"} - \text{"rate out"})\,dt$$

Average value of $f(x)$ on $[a,b]$:

$$f_{avg} = \frac{1}{b-a}\int_a^b f(x)dx = \frac{\int_a^b f(x)dx}{b-a}$$