AP Calculus

Theorems (IVT, EVT, and MVT)

Student Handout

2016-2017 EDITION

Use the following link or scan the QR code to complete the evaluation for the Study Session https://www.surveymonkey.com/r/S_SSS

Theorems (IVT, EVT, and MVT)

Students should be able to apply and have a geometric understanding of the following:

- Intermediate Value Theorem
- Mean Value Theorem for derivatives
- Extreme Value Theorem

Multiple Choice

1. (calculator not allowed)
   If f is continuous for $a \le x \le b$ and differentiable for $a < x < b$, which of the following could be false?

   (A) $f'(c) = \dfrac{f(b) - f(a)}{b - a}$ for some c such that $a < c < b$.

   (B) $f'(c) = 0$ for some c such that $a < c < b$.

   (C) f has a minimum value on $a \le x \le b$.

   (D) f has a maximum value on $a \le x \le b$.

   (E) $\int_a^b f(x)\, dx$ exists.

2. (calculator not allowed)
   The function f is defined on the closed interval $[2, 4]$ and $f(2) = f(3) = f(4)$. On the open interval $(2, 4)$, f is continuous and strictly decreasing. Which of the following statements is true?

   (A) f attains neither a minimum value nor a maximum value on the closed interval $[2, 4]$.

   (B) f attains a minimum value but does not attain a maximum value on the closed interval $[2, 4]$.

   (C) f attains a maximum value but does not attain a minimum value on the closed interval $[2, 4]$.

   (D) f attains both a minimum value and a maximum value on the closed interval $[2, 4]$.

3. (calculator not allowed)

Let f be a function with first derivative defined by $f'(x) = \frac{2x^2 - 5}{x^2}$ for $x > 0$. It is known that $f(1) = 7$ and $f(5) = 11$. What value of x in the open interval $(1, 5)$ satisfies the conclusion of the Mean Value Theorem for f on the closed interval $[1, 5]$?

(A) 1 (B) $\sqrt{\frac{5}{2}}$ (C) $\sqrt[3]{10}$ (D) $\sqrt{5}$

4. (calculator not allowed)

x	0	1	2
$f(x)$	1	k	2

The function f is continuous on the closed interval [0, 2] and has values that are given in the table above. The equation $f(x) = \frac{1}{2}$ must have at least two solutions in the interval [0, 2] if k =

(A) 0 (B) $\frac{1}{2}$ (C) 1 (D) 2 (E) 3

5. (calculator not allowed)

Let g be a continuous function on the closed interval [0, 1]. Let $g(0) = 1$ and $g(1) = 0$. Which of the following is NOT necessarily true?

(A) There exists a number h in [0, 1] such that $g(h) \geq g(x)$ for all x in [0, 1].

(B) For all a and b in [0, 1], if $a = b$, then $g(a) = g(b)$.

(C) There exists a number h in [0, 1] such that $g(h) = \frac{1}{2}$.

(D) There exists a number h in [0, 1] such that $g(h) = \frac{3}{2}$.

(E) For all h in the open interval (0, 1), $\lim_{x \to h} g(x) = g(h)$.

6. (calculator not allowed)

   If $f(x) = \sin\left(\frac{x}{2}\right)$, then there exists a number c in the interval $\frac{\pi}{2} < x < \frac{3\pi}{2}$ that satisfies the conclusion of the Mean Value Theorem. Which of the following could be c?

   (A) $\frac{2\pi}{3}$

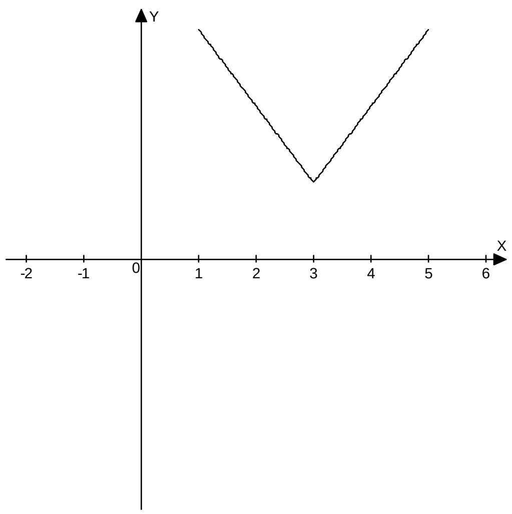
   (B) $\frac{3\pi}{4}$

   (C) $\frac{5\pi}{6}$

   (D) π

   (E) $\frac{3\pi}{2}$

7. (calculator not allowed)
   Which of the following theorems may be applied to the graph below,
   $y = |x-3| + b,\ b > 0,$ over the interval $[2,\ 4]$?



I. Mean Value Theorem
II. Intermediate Value Theorem
III. Extreme Value Theorem

(A) I only (B) II only (C) III only (D) II and III only (E) I, II, and III

8. (calculator not allowed)
   The function f is defined by $f(x) = 4x^2 - 5x + 1$. The application of the Mean Value Theorem to f on the interval $0 < x < 2$ guarantees the existence of a value c, where $0 < c < 2$ such that $f'(c) =$

   (A) 1 (B) 3 (C) 7 (D) 8

9. (calculator not allowed)
   A function of f is continuous on the closed interval $[2,5]$ with $f(2) = 17$ and $f(5) = 17$. Which of the following additional conditions guarantees that there is a number c in the open interval $(2,5)$ such that $f'(c) = 0$?

   (A) No additional conditions are necessary
   (B) f has a relative extremum on the open interval $(2,5)$.
   (C) f is differentiable on the open interval $(2,5)$.
   (D) $\int_2^5 f(x)dx$ exists

10. (calculator not allowed)

x	0	2	4	8
$f(x)$	3	4	9	13
$f'(x)$	0	1	1	2

   The table above gives values of a differentiable function f and its derivatives at selected values of x. If h is the function given by $h(x) = f(2x)$, which of the following statements must be true?

   (I) h is increasing on $2 < x < 4$.
   (II) There exists c, where $0 < c < 4$, such that $h(c) = 12$.
   (III) There exists c, where $0 < c < 2$, such that $h'(c) = 3$.

   (A) II only
   (B) I and III only
   (C) II and III only
   (D) I, II, and III

11. (calculator allowed)
The function f is continuous for $-2 \le x \le 1$ and differentiable for $-2 < x < 1$. If $f(-2) = -5$ and $f(1) = 4$, which of the following statements could be false?

(A) There exists c, where $-2 < c < 1$, such that $f(c) = 0$.

(B) There exists c, where $-2 < c < 1$, such that $f'(c) = 0$.

(C) There exists c, where $-2 < c < 1$, such that $f(c) = 3$.

(D) There exists c, where $-2 < c < 1$, such that $f'(c) = 3$.

(E) There exists c, where $-2 \le c \le 1$ such that $f(c) \ge f(x)$ for all x on the closed interval $-2 \le x \le 1$.

12. (calculator not allowed)

x	0	2	5	9	11
$g(x)$	1	2.8	1.7	1	3.4

The table above shows selcted values of a continuous function g. For $0 \le x \le 11$, what is the fewest possible number of times $g(x) = 2$?

(A) One (B) Two (C) Three (D) Four

Free Response

13. (calculator not allowed)

t (minutes)	0	1	2	3	4	5	6
$C(t)$ (ounces)	0	5.3	8.8	11.2	12.8	13.8	14.5

Hot water is dripping through a coffeemaker, filling a large cup with coffee. The amount of coffee in the cup at time t, $0 \le t \le 6$, is given by a differentiable function C, where t is measured in minutes. Selected values of $C(t)$, measured in ounces, are given in the table above.

Is there a time t, $2 \le t \le 4$, at which $C'(t) = 2$? Justify your answer.

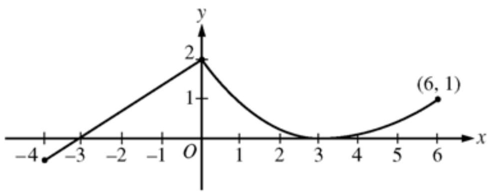
14. (calculator not allowed)

t (minutes)	0	2	5	8	12
$v_A(t)$ (meters/minute)	0	100	40	-120	-150

Train A runs back and forth on an east-west section of railroad track. Train A's velocity, measured in meters per minute, is given by a differentiable function $v_A(t)$, where time t is measured in minutes. Selected values for $v_A(t)$ are given in the table above.

(b) Do the data in the table support the conclusion that train A's velocity is -100 meters per minute at some time t with $5 < t < 8$? Give a reason for your answer.

15. (calculator allowed)

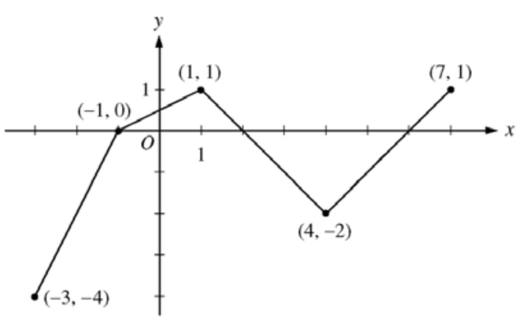


Graph of f

A continuous function f is defined on the closed interval $-4 \le x \le 6$. The graph of f consists of a line segment and a curve that is tangent to the x-axis at $x = 3$, as shown in the figure above. On the interval $0 < x < 6$, the function f is twice differentiable, with $f''(x) > 0$.

(c) Is there a value of a, $-4 \le a < 6$, for which the Mean Value Theorem, applied to the interval $[a, 6]$, guarantees a value c, $a < c < 6$, at which $f'(c) = \frac{1}{3}$? Justify your answer.

16. (calculator not allowed)



Graph of g'

Let g be a continuous function with $g(2) = 5$. The graph of the piecewise-linear function g', the derivative of g, is shown above for $-3 \le x \le 7$.

(d) Find the average rate of change of $g'(x)$ on the interval $-3 \le x \le 7$. Does the Mean Value Theorem applied on the interval $-3 \le x \le 7$ guarantee a value of c, for $-3 < c < 7$, such that $g''(c)$ is equal to this average rate of change? Why or why not?

17. (calculator allowed)

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
1	6	4	2	5
2	9	2	3	1
3	10	-4	4	2
4	-1	3	6	7

The functions f and g are differentiable for all real numbers, and g is strictly increasing. The table above gives values of the functions and their first derivatives at selected values of x. The function h is given by $h(x) = f(g(x)) - 6$.

(a) Explain why there must be a value r for $1 < r < 3$ such that $h(r) = -5$.

(b) Explain why there must be a value c for $1 < c < 3$ such that $h'(c) = -5$.

18. (calculator not allowed)

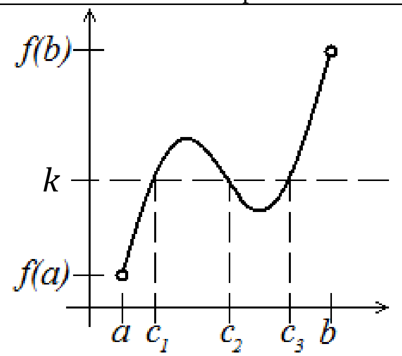
Let f be a twice-differentiable function such that $f(2)=5$ and $f(5)=2$. Let g be the function given by $g(x)=f\big(f(x)\big)$.

(a) Explain why there must be a value c for $2 < c < 5$ such that $f'(c)=-1$.

(b) Show that $g'(2)=g'(5)$. Use this result to explain why there must be a value k for $2<k<5$ such that $g''(k)=0$.

(d) Let $h(x)=f(x)-x$. Explain why there must be a value r for $2 < r < 5$ such that $h(r)=0$.

Reference Page

<table>
<tr><th>Name</th><th>Formal Statement</th><th>Restatement</th><th>Graph</th><th>Notes</th></tr>
<tr>
<td>IVT</td>
<td>If $f(x)$ is continuous on a closed interval $[a, b]$ and k is any number between $f(a)$ and $f(b)$, then there exists at least one value c in $[a, b]$ such that $f(c)=k$.</td>
<td>On a continuous function, you will hit every y-value between two given y-values at least once.</td>
<td>
</td>
<td>When writing a justification using the IVT, you must state the function is continuous even if this information is provided in the question.</td>
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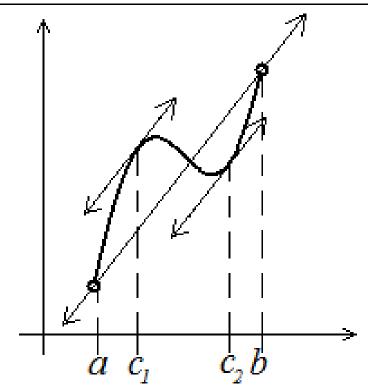
<tr>
<td>MVT</td>
<td>If $f(x)$ is continuous on the closed interval $[a, b]$ and differentiable on (a, b), then there must exist at least one value c in (a, b) such that $$f'(c)=\frac{f(b)-f(a)}{b-a}$$</td>
<td>If conditions are met (very important!) there is at least one point where the slope of the tangent line equals the slope of the secant line.</td>
<td>
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<td>When writing a justification using the MVT, you must state the function is differentiable (continuity is implied by differentiability) even if this information is provided in the question.

(Questions may ask students to justify why the MVT cannot be applied often using piecewise functions that are not differentiable over an open interval.)</td>
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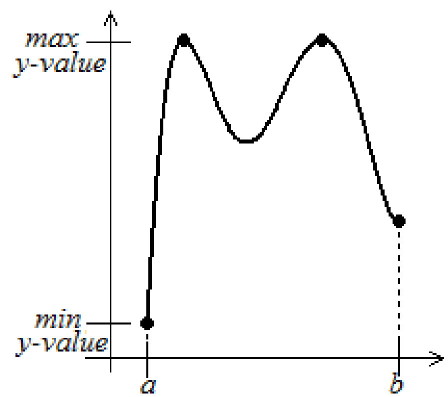
<tr>
<td>EVT</td>
<td>A continuous function $f(x)$ on a closed interval $[a, b]$ attains both an absolute maximum $f(c) \geq f(x)$ for all x in the interval and an absolute minimum $f(c) \leq f(x)$ for all x in the interval</td>
<td>Every continuous function on a closed interval has a highest y-value and a lowest y-value.</td>
<td>
</td>
<td>When writing a justification using the EVT, you must state the function is continuous on a closed interval even if this information is provided in the question.</td>
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