AP Calculus

Analyzing a Function Based on its Derivatives

Student Handout

2016-2017 EDITION

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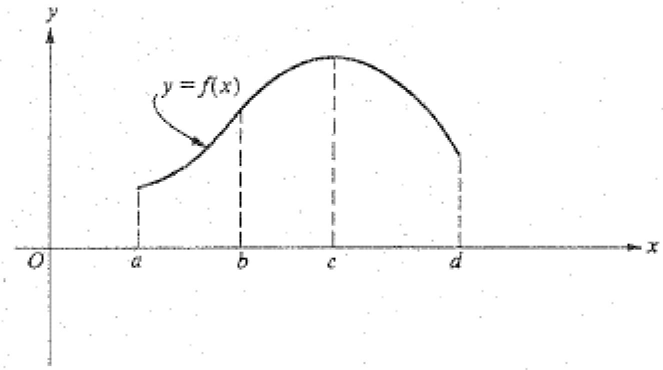
Analyzing a Function Based on its Derivatives

Students need to be able to:

- Locate critical numbers of the function and its derivatives.
- Determine the graph of the function given the graph of its derivative and vice versa.
- Determine whether a function is increasing or decreasing using information about the derivative.
- Determine the concavity of a function's graph using information about the first or second derivative.
- Locate a function's relative and absolute extrema from its derivative.
- Locate a function's point(s) of inflection from its first or second derivative.
- Reason from a graph without finding an explicit rule that graph represents.
- Sketch any of the related functions.
- Write justifications and explanations.
  - Must be written in sentence form.
  - Avoid using the pronoun "it" when justifying extrema.
  - Use "the function *f* (or appropriate name of the function as given)," "the derivative of *f*," or "the second derivative of *f* " instead of "the graph" or "the slope" in explanations.

Multiple Choice

1. (calculator not allowed)

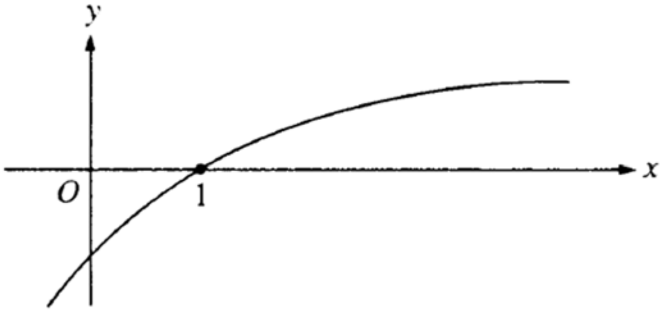


The graph of $y = f(x)$ is shown in the figure above. On which of the following intervals are $\frac{dy}{dx} > 0$ and $\frac{d^2y}{dx^2} < 0$?

I. $a < x < b$
II. $b < x < c$
III. $c < x < d$

(A) I only (B) II only (C) III only (D) I and II (E) II and III

2. (calculator not allowed)



The graph of a twice-differentiable function f is shown in the figure above. Which of the following is true?

(A) $f(1) < f'(1) < f''(1)$
(B) $f(1) < f''(1) < f'(1)$
(C) $f'(1) < f(1) < f''(1)$
(D) $f''(1) < f(1) < f'(1)$
(E) $f''(1) < f'(1) < f(1)$

3. (calculator not allowed)
The function f is given by $f\left(x\right)=x^4+x^2-2$. On which of the following intervals is f increasing?

(A) $\left(-\frac{1}{\sqrt{2}},\infty\right)$
(B) $\left(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)$
(C) $(0,\ \infty)$
(D) $(-\infty,\ 0)$
(E) $\left(-\infty,-\frac{1}{\sqrt{2}}\right)$

4. (calculator not allowed)
Given the function defined by $f(x)=3x^5-20x^3$, find all values of x for which the graph of f is concave up.

(A) $x>0$
(B) $-\sqrt{2}<x<0$ or $x>\sqrt{2}$
(C) $-2<x<0$ or $x>2$
(D) $x>\sqrt{2}$
(E) $-2<x<2$

5. (calculator not allowed)
What are all values of x for which the function f defined by $f(x)=\left(x^2-3\right)e^{-x}$ is increasing?

(A) There are no such values of x.
(B) $x<-1$ and $x>3$
(C) $-3<x<1$
(D) $-1<x<3$
(E) All values of x

6. (calculator not allowed)
If $f''(x) = x(x+1)(x-2)^2$, then the graph of f has inflection points when $x =$

(A) -1 only
(B) 2 only
(C) -1 and 0 only
(D) -1 and 2 only
(E) -1, 0, and 2 only

7. (calculator not allowed)

x	-4	-3	-2	-1	0	1	2	3	4
$g'(x)$	2	3	0	-3	-2	-1	0	3	2

The derivative g' of a function g is continuous and has exactly two zeros. Selected values of g' are given in the table above. If the domain of g is the set of all real numbers, then g is decreasing on which of the following intervals?

(A) $-2 \le x \le 2$ only
(B) $-1 \le x \le 1$ only
(C) $x \ge -2$ only
(D) $x \ge 2$ only
(E) $x \le -2$ or $x \ge 2$

8. (calculator not allowed)
The function $y = f(x)$ is differentiable and decreasing for all real numbers. On what interval is $y = f(x^2 - 4x)$ decreasing?

(A) $[0, 4]$
(B) $(-\infty, 2]$
(C) $[2, \infty)$
(D) $(-\infty, \infty)$

9. (calculator allowed)
For all x in the closed interval $[2, 5]$, the function f has a positive first derivative and a negative second derivative. Which of the following could be a table for f ?

(A)

x	2	3	4	5
$f(x)$	7	9	12	16

(B)

x	2	3	4	5
$f(x)$	7	11	14	16

(C)

x	2	3	4	5
$f(x)$	16	12	9	7

(D)

x	2	3	4	5
$f(x)$	16	14	11	7

(E)

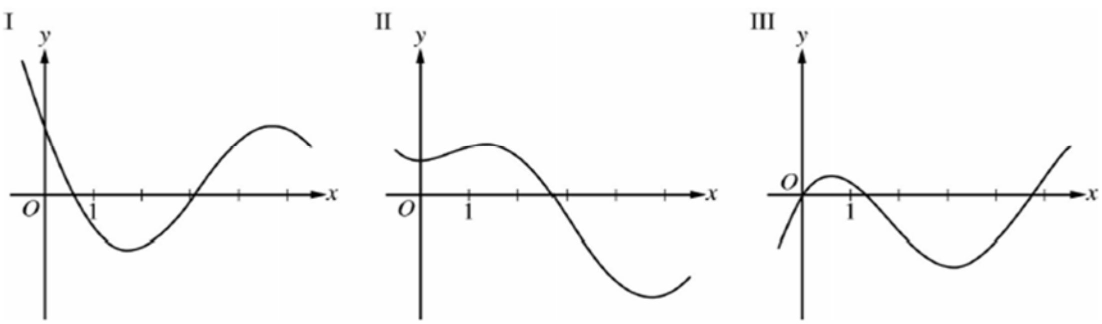
x	2	3	4	5
$f(x)$	16	13	10	7

10. (calculator not allowed)
Let f be the function defined by $f(x) = x^3 - 3x^2$. On which of the following intervals is f' both positive and decreasing?
(A) $(-\infty, 0)$
(B) $(-\infty, 1)$
(C) $(0, 2)$
(D) $(1, 2)$

11. (calculator not allowed)
If $f(x) = x^2 e^x$, then the graph of f is decreasing for all x such that

(A) $x < -2$ (B) $-2 < x < 0$ (C) $x > -2$ (D) $x < 0$ (E) $x > 0$

12. (calculator not allowed)



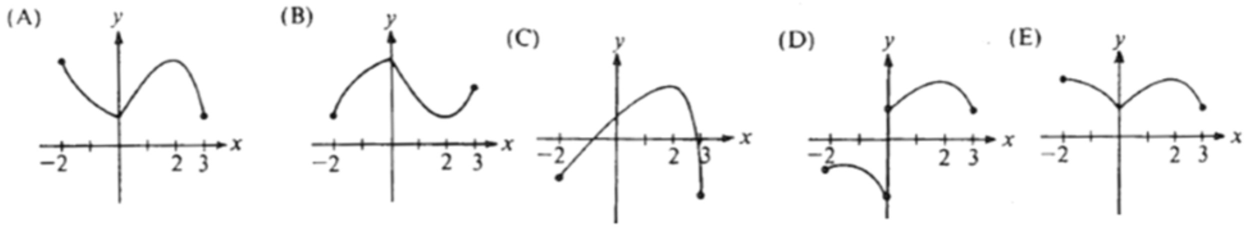
Three graphs labeled I, II, III are shown above. One is the graph of f, one is the graph of f', and one is the graph of f''. Which of the following correctly identifies each of the three graphs?

	f	f'	f''
(A)	I	II	III
(B)	II	I	III
(C)	II	III	I
(D)	III	I	II

13. (calculator allowed)
Let f be the function given by $f(x) = \cos(2x) + \ln(3x)$. What is the least value of x at which the graph of f changes concavity?

(A) 0.56 (B) 0.93 (C) 1.18 (D) 2.38 (E) 2.44

14. (calculator not allowed)
Let f be a function that is continuous on the closed interval $[-2, 3]$ such that $f'(0)$ does not exist, $f'(2)=0$, and $f''(x)<0$ for all x except $x=0$. Which of the following could be the graph of f ?



15. (calculator allowed)
Let f be the function with derivative defined by $f'(x)=\sin\left(x^3\right)$ on the interval $-1.8<x<1.8$. How many points of inflection does the graph of f have on this interval?

(A) Two (B) Three (C) Four (D) Five (E) Six

16. (calculator allowed)
If the derivative of f is given by $f'(x)=e^x-3x^2$, at which of the following values of x does f have a relative maximum value?

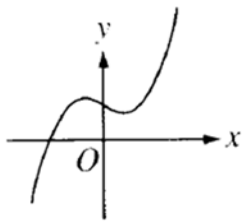
(A) –0.46
(B) 0.20
(C) 0.91
(D) 0.95
(E) 3.73

17. (calculator allowed)

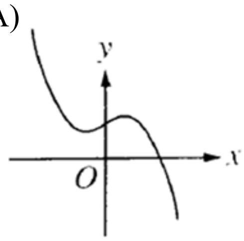
If g is a differentiable function such that $g(x) < 0$ for all real numbers x and if $f'(x) = (x^2 - 4)g(x)$, which of the following is true?

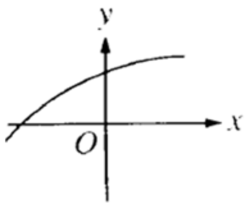
(A) f has a relative maximum at $x = -2$ and a relative minimum at $x = 2$.
(B) f has a relative minimum at $x = -2$ and a relative maximum at $x = 2$.
(C) f has relative minima at $x = -2$ and at $x = 2$.
(D) f has relative maxima at $x = -2$ and at $x = 2$.
(E) It cannot be determined if f has any relative extrema.

18. (calculator not allowed)



The graph $y = h(x)$ is shown above. Which of the following could be the graph of $y = h'(x)$?

(A)


(B)


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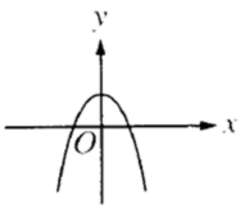
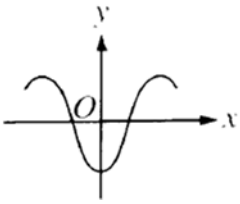
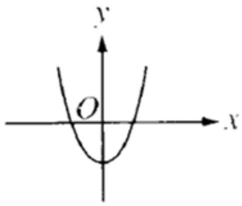
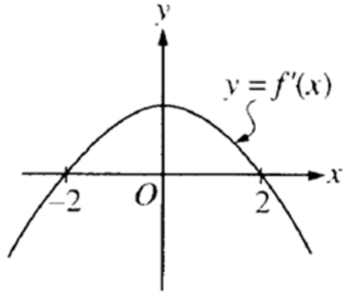


(D)


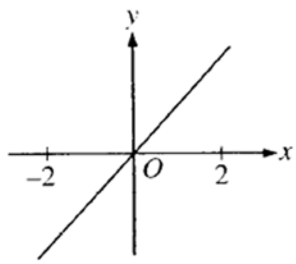
(E)


19. (calculator not allowed)

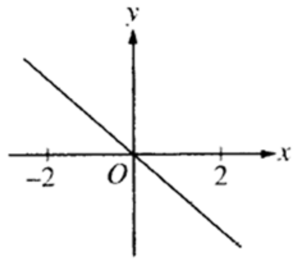


The graph of the derivative of f is shown in the figure above. Which of the following could be the graph of f ?

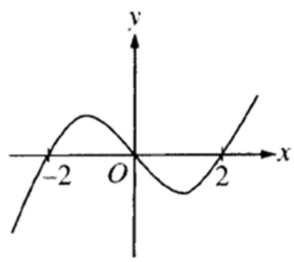
(A)



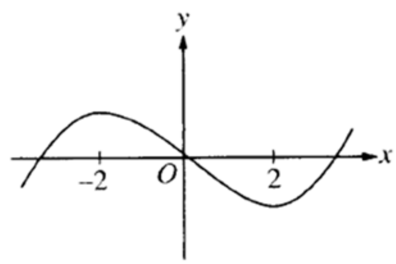
(B)



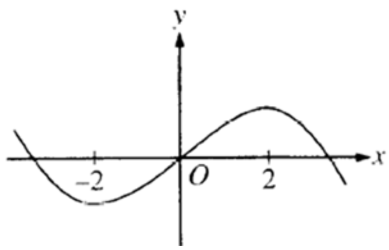
(C)



(D)



(E)

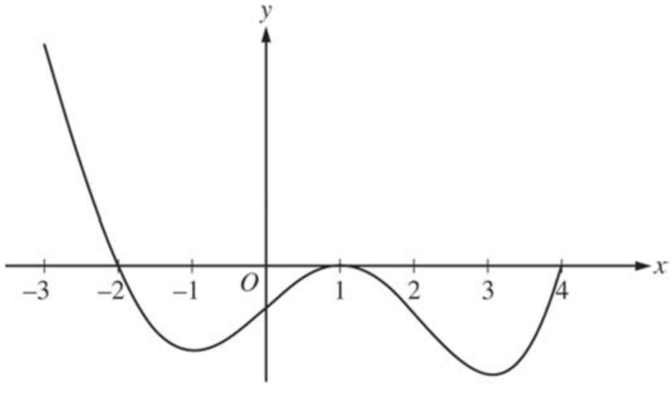


20. (calculator not allowed)
The function f is given by $f(x) = x^3 + 3x^2 - 9x - 4$. What is the absolute maximum value of f on the closed interval $[-4, 2]$?

(A) –3 (B) 2 (C) 15 (D) 23

Free Response

21. (calculator not allowed)



Graph of f'

The figure above shows the graph of f', the derivative of a twice-differentiable function f, on the interval $[-3, 4]$. The graph of f' has horizontal tangents at $x = -1$, $x = 1$, and $x = 3$. The areas of the regions bounded by the *x*-axis and the graph of f' on the intervals $[-2, 1]$ and $[1, 4]$ are 9 and 12, respectively.

(a) Find all the *x*-coordinates at which f has a relative maximum. Give a reason for your answer.

(b) On what open intervals contained in $-3 < x < 4$ is the graph of f both concave down and decreasing?

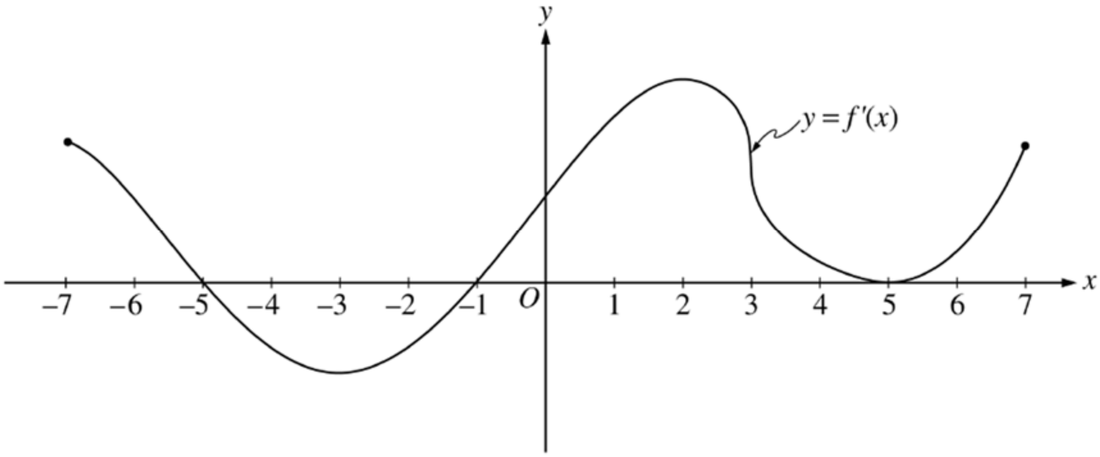
(c) Find the *x*-coordinates of all points of inflection for the graph of f, Give a reason for your answer.

22. (calculator not allowed)
Consider a differentiable function f having domain all positive real numbers, and for which it is known that $f'(x) = (4 - x)x^{-3}$ for $x > 0$.

(a) Find the x-coordinate of the critical point of f. Determine whether the point is a relative maximum, a relative minimum, or neither for the function f. Justify your answer.

(b) Find all intervals on which the graph of f is concave down. Justify your answer.

23. (calculator allowed)



The figure above shows the graph of f', the derivative of f, for $-7 \le x \le 7$. The graph of f' has horizontal tangent lines at $x = -3$, $x = 2$, and $x = 5$, and a vertical tangent line at $x = 3$.

(a) Find all values of x, for $-7 < x < 7$, at which f attains a relative minimum. Justify your answer.

(b) Find all values of x, for $-7 < x < 7$, at which f attains a relative maximum. Justify your answer.

(c) Find all values of x, for $-7 < x < 7$, at which $f'' < 0$.

Analyzing f, f', and f'' Reference Page

Free response questions on this topic often require students to justify their answers. Students may use number lines to analyze the characteristics of the function; however, the justification must be written in sentence form. This justification should avoid using a pronoun such as "it" to describe the function and should make use of calculus involving the first or second derivative test or theorems.

Increase/Decrease

- If $f'(x) > 0$ on an interval, then $f(x)$ is increasing on the interval.
- If $f'(x) < 0$ on an interval, then $f(x)$ is decreasing on the interval.

Relative or Local Extrema – highest or lowest point in the neighborhood

- First derivative test
  - Candidates – critical numbers (x-values that make f' zero or undefined where f is defined)
  - Test – (1) set up an f' number line; label with candidates
    (2) test each section to see if f' is positive or negative
    (3) relative maximum occurs when f' changes from + to –
    relative minimum occurs when f' changes from – to +
- Second derivative test
  - Candidates – critical numbers (x-values that make f' zero or undefined where f is defined)
  - Test – (1) substitute each critical number into the second derivative
    (2) $f'' > 0$, relative minimum
    $f'' < 0$, relative maximum
    (3) $f'' = 0$, the test fails

Absolute or Global Extrema – highest or lowest point in the domain

- Absolute Extrema Test
  - Candidates – critical numbers and endpoints of the domain
  - Test – (1) find the y-values for each candidate
    (2) the absolute maximum value is the largest y-value,
    the absolute minimum value is the smallest y-value

Concavity

- If $f''(x) > 0$ (or $f'(x)$ is increasing) on an interval, then $f(x)$ is concave up on that interval.
- If $f''(x) < 0$ (or $f'(x)$ is decreasing) on an interval, then $f(x)$ is concave down on that interval.

Point of inflection – point where the concavity changes

- Determining points of inflection using the first derivative
  - The graph of f has a point of inflection where f' changes from increasing to decreasing or decreasing to increasing.
- Determining points of inflection using the second derivative
  - Candidates – x-values for which f'' is zero or undefined where f is defined
  - Test – (1) set up an f'' number line; label with candidates
    (2) test each section to see if f'' is positive or negative
    (3) any change in the sign of f'' indicates a point of inflection

Justifications for Increasing/Decreasing Intervals of a function

Remember: $f'(x)$ determines whether a function is increasing or decreasing, so always use the sign of $f'(x)$ when determining and justifying whether a function $f(x)$ is increasing or decreasing on (a, b).

Situation	Explanation
$f(x)$ is increasing on the interval (a, b)	$f(x)$ is increasing on the interval (a, b) because $f'(x) > 0$
$f(x)$ is decreasing on the interval (a, b)	$f(x)$ is decreasing on the interval (a, b) because $f'(x) < 0$

Justifications of Relative Minimums/Maximums and Points of Inflection

Sign charts are very commonly used in calculus classes and are a valuable tool for students to use when testing for relative extrema and points of inflection. However, a sign chart will never earn students any points on the AP exam. Students should use sign charts when appropriate to help make determinations, but they cannot be used as a justification or explanation on the exam.

Situation (at a point $x = a$ on the function $f(x)$)	Proper Explanation/Reasoning
Relative Minimum	$f(x)$ has a relative minimum at the point $x = a$ because $f'(x)$ changes signs from negative to positive when $x = a$.
Relative Maximum	$f(x)$ has a relative maximum at the point $x = a$ because $f'(x)$ changes signs from positive to negative when $x = a$.
Point of Inflection	$f(x)$ has a point of inflection at the point $x = a$ because $f''(x)$ changes sign when $x = a$