NATIONAL MATH + SCIENCE INITIATIVE

AP Calculus

Limits, Continuity, and Differentiability

Student Handout

2016-2017 EDITION

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Limits, Continuity, and Differentiability

Students should be able to:

- Determine limits from a graph
- Know the relationship between limits and asymptotes (i.e., limits that become infinite at a finite value or finite limits at infinity)
- Compute limits algebraically
- Discuss continuity algebraically and graphically and know its relation to limit.
- Discuss differentiability algebraically and graphically and know its relation to limits and continuity
- Recognize the limit definition of derivative and be able to identify the function involved and the point at which the derivative is evaluated. For example, since $f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$, recognize that $\lim_{h\to 0}\frac{\cos(\pi+h)-\cos(\pi)}{h}$ is simply the derivative of cos(x) at $x=\pi$.
- L'Hôspital's Rule

Multiple Choice

1. (calculator not allowed)
   $\lim_{n\to\infty}\frac{4n^2}{n^2+10000n}$ is

   (A) 0
   (B) $\frac{1}{2500}$
   (C) 1
   (D) 4
   (E) nonexistent

2. (calculator not allowed)
   The $\lim_{h\to 0}\frac{\tan 3(x+h)-\tan 3x}{h}$ is

   (A) 0
   (B) $3\sec^2(3x)$
   (C) $\sec^2(3x)$
   (D) $3\cot(3x)$
   (E) nonexistent

3. (calculator not allowed)

$$\lim_{x\to 0} \frac{7x - \sin x}{x^2 + \sin(3x)} =$$

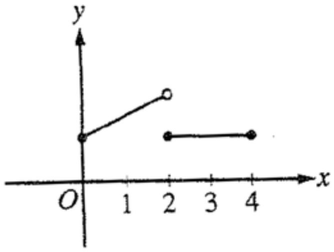
(A) 6
(B) 2
(C) 1
(D) 0

4. (calculator not allowed)

At $x = 3$, the function given by $f(x) = \begin{cases} x^2, & x < 3 \\ 6x - 9, & x \geq 3 \end{cases}$ is

(A) undefined.
(B) continuous but not differentiable.
(C) differentiable but not continuous.
(D) neither continuous nor differentiable.
(E) both continuous and differentiable

5. (calculator allowed)



Graph of f

The figure above shows the graph of a function f with domain $0 \leq x \leq 4$. Which of the following statements are true?

I. $\lim_{x\to 2^-} f(x)$ exists

II. $\lim_{x\to 2^+} f(x)$ exists

III. $\lim_{x\to 2} f(x)$ exists

(A) I only (B) II only (C) I and II only (D) I and III only (E) I, II, and III

6. (calculator not allowed)

   If $f(x) = 2x^2 + 1$, then $\lim_{x \to 0} \frac{f(x) - f(0)}{x^2}$ is

   (A) 0
   (B) 1
   (C) 2
   (D) 4
   (E) nonexistent

7. (calculator not allowed)

   If $f'(x) = \cos x$ and $g'(x) = 1$ for all x, and if $f(0) = g(0) = 0$, then $\lim_{x \to 0} \frac{f(x)}{g(x)}$ is

   (A) $\frac{\pi}{2}$

   (B) 1

   (C) 0

   (D) −1

   (E) nonexistent

8. (calculator not allowed)

   $$f(x) = \begin{cases} \ln(4x - 7) & \text{if } x < 2 \\ 4x - 7 & \text{if } x \geq 2 \end{cases}$$

   Let f be the function defined above. Which of the following statements about f are true?

   I. $\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x)$
   II. $\lim_{x \to 2^-} f'(x) = \lim_{x \to 2^+} f'(x)$
   III. f is differentiable at $x = 2$

   (A) I only
   (B) II only
   (C) II and III only
   (D) I, II, and III

9. (calculator not allowed)
   If $\lim_{x \to a} f(x) = L$ where L is a real number, which of the following must be true?

   (A) $f'(a)$ exists.
   (B) $f(x)$ is continuous at $x = a$.
   (C) $f(x)$ is defined at $x = a$.
   (D) $f(a) = L$
   (E) None of the above

10. (calculator not allowed)
    For $x \geq 0$, the horizontal line $y = 2$ is an asymptote for the graph of the function f. Which of the following statements must be true?

    (A) $f(0) = 2$
    (B) $f(x) \neq 2$ for all $x \geq 0$
    (C) $f(2)$ is undefined.
    (D) $\lim_{x \to 2} f(x) = \infty$
    (E) $\lim_{x \to \infty} f(x) = 2$

11. (calculator not allowed)
    If the graph of $y = \frac{ax+b}{x+c}$ has a horizontal asymptote at $y = 2$ and a vertical asymptote at $x = -3$, then $a + c =$

    (A) -5
    (B) -1
    (C) 0
    (D) 1
    (E) 5

12. (calculator not allowed)
    $\lim_{x \to \infty} \frac{(2x-1)(3-x)}{(x-1)(x+3)}$ is

    (A) -3
    (B) -2
    (C) 2
    (D) 3
    (E) nonexistent

13. (calculator not allowed)

$$f(x)=\begin{cases}\dfrac{x^2-4}{x-2}, & x\neq 2\\ 1, & x=2\end{cases}$$

Let f be the function defined above. Which of the following statements about f are true?

I. f has a limit at $x=2$.
II. f is continuous at $x=2$.
III. f is differentiable at $x=2$.

(A) I only
(B) II only
(C) III only
(D) I and II only
(E) I, II, and III

$$f(x)=\begin{cases}x+2b \text{ if } x\leq 2\\ ax^2 \text{ if } x>2\end{cases}$$

14. Let f be the function given above. What are all values of a and b for which f is differentiable at $x=2$?

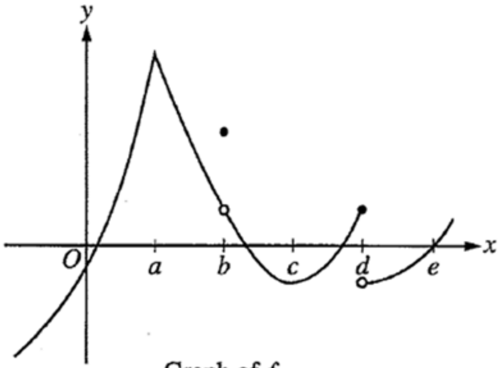
(A) $a=\frac{1}{4}$ and $b=-\frac{1}{2}$
(B) $a=\frac{1}{4}$ and $b=\frac{1}{2}$
(C) $a=\frac{1}{4}$ and b is any real number
(D) $a=b+2$, where b is any real number
(E) There are no such values of a and b

15. (calculator not allowed)

If the function f is continuous for all real numbers and if $f(x)=\dfrac{x^2-4}{x+2}$ when $x\neq -2$, then $f(-2)=$

(A) -4
(B) -2
(C) -1
(D) 0
(E) 2

16. (calculator not allowed)

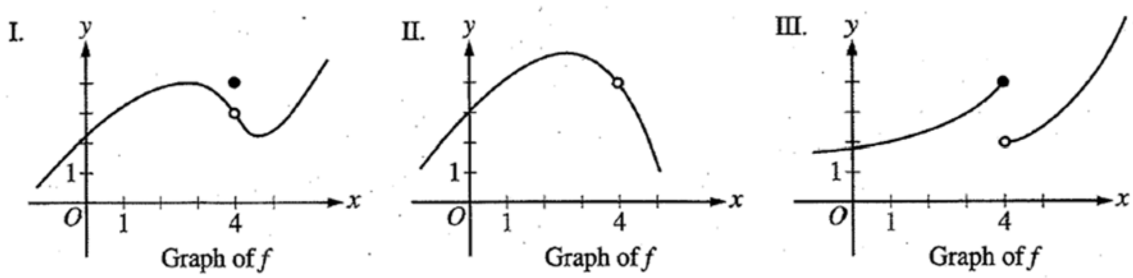


Graph of f

The graph of a function f is shown above. At which value of x is f continuous, but not differentiable?

(A) a
(B) b
(C) c
(D) d
(E) e

17. (calculator allowed)



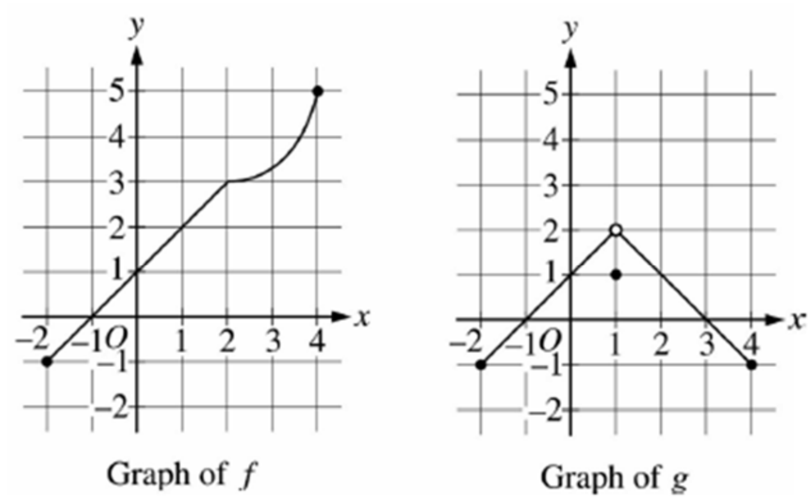
For which of the following does $\lim_{x \to 4} f(x)$ exist?

(A) I only
(B) II only
(C) III only
(D) I and II only
(E) I and III only

18. (calculator not allowed)

$\lim_{x \to e} \frac{\ln x - 1}{x - e}$ is

(A) $\frac{1}{e}$
(B) 1
(C) e
(D) nonexistent



19. The graphs of the functions f and g are shown above. The value of $\lim_{x \to 1} f(g(x))$ is

(A) 1
(B) 2
(C) 3
(D) nonexistent

20. (calculator not allowed)

Let f be a function defined by $f(x)=\begin{cases}1-2\sin x, & x\le 0\\ e^{-4x}, & x>0.\end{cases}$

(a) Show that f is continuous at $x=0$.

21. (calculator not allowed) 2012 AB 4

The function f is defined by $f(x)=\sqrt{25-x^2}$ for $-5\le x\le 5$.

(c) Let g be the function defined by $g(x)=\begin{cases}f(x) & \text{for } -5\le x\le -3\\ x+7 & \text{for } -3<x<5.\end{cases}$

Is g continuous at $x=-3$? Use the definition of continuity to explain your answer.

2012 AB 4 Rubric

The function f is defined by $f(x)=\sqrt{25-x^2}$ for $-5 \le x \le 5$.

(c) Let g be the function defined by $g(x)=\begin{cases} f(x) & \text{for } -5 \le x \le -3 \\ x+7 & \text{for } -3 < x < 5. \end{cases}$

Is g continuous at $x=-3$? Use the definition of continuity to explain your answer.

(c) $\lim_{x\to -3^-} g(x) = \lim_{x\to -3^-} f(x) = \lim_{x\to -3^-} \sqrt{25-x^2} = 4$ $\lim_{x\to -3^+} g(x) = \lim_{x\to -3^+} (x+7) = 4$ Therefore, $\lim_{x\to -3} g(x) = 4$. $g(-3) = f(-3) = 4$ So, $\lim_{x\to -3} g(x) = g(-3)$.	2 { 1: considers one-sided limits 1: answer with explanation

Student Samples:

4 4 4 4 4 4 4 4 4 4

NO CALCULATOR ALLOWED

4A₂

(c) Let g be the function defined by $g(x)=\begin{cases} f(x) & \text{for } -5 \le x \le -3 \\ x+7 & \text{for } -3 < x \le 5. \end{cases}$

Is g continuous at $x=-3$? Use the definition of continuity to explain your answer.

$f(x) = \sqrt{25-x^2}$

$\lim_{x\to -3^-} g(x)$
$= \lim_{x\to -3} \left(\sqrt{25-x^2}\right)$
$= \sqrt{25-(-3)^2} = \sqrt{25-9}$
$= \sqrt{16} = 4$

$g(-3)$
$= \sqrt{25-(-3)^2} = \sqrt{25-9}$
$= \sqrt{16} = 4.$

$\lim_{x\to -3^+} g(x)$
$= \lim_{x\to -3} (x+7)$
$= (-3+7) = 4$

Yes g is continuous at $x=-3$. The left-and right-hand limits of g ~~are~~ ~~equal~~ as x approaches -3 are both equal to the definition of g at $x=-3$.

yond this border.

4 4 4 4 4 4 4 4 4 4 4B2

NO CALCULATOR ALLOWED

(c) Let g be the function defined by $g(x) = \begin{cases} f(x) & \text{for } -5 \le x \le -3 \\ x+7 & \text{for } -3 < x \le 5. \end{cases}$

Is g continuous at $x = -3$? Use the definition of continuity to explain your answer.

$\lim_{x \to -3^+} x+7 = 4$

$\lim_{x \to -3^-} \sqrt{25-x^2} = 4$

Since the $\lim_{x \to -3^+} g(x) = \lim_{x \to -3^-} g(x)$, the $\lim_{x \to -3} g(x)$ exists. Since $\lim_{x \to -3} g(x) = g(-3)$ and $g(-3)$ exists, g is continuous at $x=-3$

4 4 4 4 4 4 4 4 4 4 4C2

NO CALCULATOR ALLOWED

(c) Let g be the function defined by $g(x) = \begin{cases} f(x) & \text{for } -5 \le x \le -3 \\ x+7 & \text{for } -3 < x \le 5. \end{cases}$

Is g continuous at $x = -3$? Use the definition of continuity to explain your answer.

$\sqrt{25-x^2} = x+7$

$\sqrt{25-(-3)^2} = (-3)+7$

$4 = 4$

g is continuous at $x=-3$ because the functions that meet at $x=-3$ have the same value at $x=-3$.

$f(-3) = (-3)+7$

(c) Let g be the function defined by $g(x) = \begin{cases} f(x) & \text{for } -5 \le x \le -3 \\ x+7 & \text{for } -3 < x \le 5. \end{cases}$

Is g continuous at $x = -3$? Use the definition of continuity to explain your answer.

4 D_2

$\sqrt{25-(-3)^2} = (-3)+7$

$4 = 4$

g is continuous at x=-3 because the y value at x=-3 for both functions is 4. This shows that the function doesn't have to break when transitioning from f(x) to x+7, and that $\lim_{x \to -3^+} g(x) = \lim_{x \to -3^-} g(x)$.

(c) Let g be the function defined by $g(x) = \begin{cases} f(x) & \text{for } -5 \le x \le -3 \\ x+7 & \text{for } -3 < x \le 5. \end{cases}$

Is g continuous at $x = -3$? Use the definition of continuity to explain your answer.

4 E_2

$f(-3) = 4$

$x+7 \Rightarrow (-3)+7 = 4$ } ✓

yes, g is continuous at x=-3

Limits, Continuity, and Differentiability Reference Page

Existence of a Limit at a Point

A function $f(x)$ has a limit L as x approaches c if and only if the left-hand and right-hand limits at c exist and are equal.

1. $\lim\limits_{x \to c^-} f(x)$ exists
2. $\lim\limits_{x \to c^+} f(x)$ exists
3. $\lim\limits_{x \to c^-} f(x) = \lim\limits_{x \to c^+} f(x)$ $\quad\quad\therefore \lim\limits_{x \to c} f(x) = L$

Continuity

A function is continuous on an interval if it is continuous at every point of the interval. Intuitively, a function is continuous if its graph can be drawn without ever needing to pick up the pencil. This means that the graph of $y = f(x)$ has no "holes", no "jumps" and no vertical asymptotes at $x = a$. When answering free response questions on the AP exam, the formal definition of continuity is required. To earn all of the points on the free response question scoring rubric, all three of the following criteria need to be met, with work shown:

A function is *continuous* at a point $x = a$ if and only if:

1. $f(a)$ exists
2. $\lim\limits_{x \to a} f(x)$ exists
3. $\lim\limits_{x \to a} f(x) = f(a)$ (i.e., the limit equals the function value)

Limit Definition of a Derivative

The derivative of a function $f(x)$ with respect to x is the function $f'(x)$ whose value at x is

$f'(x) = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$, provided the limit exists.

Alternative Form for Definition of a Derivative

The derivative of a function $f(x)$ at $x = a$ is $f'(a) = \lim\limits_{x \to a} \dfrac{f(x) - f(a)}{x - a}$, provided the limit exists.

Continuity and Differentiability

Differentiability implies continuity (but not necessarily vice versa) If a function is differentiable at a point (at every point on an interval), then it is continuous at that point (on that interval). The converse is not always true: continuous functions may not be differentiable. It is possible for a function to be continuous at a specific value for a but not differentiable there.

L'Hospital's Rule (returns on 2017 AB exam)

Given that f and g are differentiable functions on an open interval (a, b) containing c (except possibly at c itself), assume that $g'(x) \neq 0$ for all x in the interval (except possibly at c itself). If $\lim_{x \to c} \frac{f(x)}{g(x)}$ produces the indeterminate form $\frac{0}{0}$, then $\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$ provided the limit on the right exists (or is infinite). This result also applies when the limit produces any one of the indeterminate forms $\frac{\infty}{\infty}, \frac{-\infty}{\infty}, \frac{\infty}{-\infty},$ or $\frac{-\infty}{-\infty}$.